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REVISED AND ENLARGED COLLECTION OF PLASMA PHYSICS FORMULAS AND --ETC(U)

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Revised and Enlarged Collection of Plasma Physics Formulas and Data

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May 1977

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This report is intended to serve as a kind of users' manual for the NRL Plasma Formulary, supplying explanations worked examples and references, for which space in the pocket version was insufficient. A certain amount of supplementary material has been included.

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CONTENTS

REVISED AND ENLARGED COLLECTION OF PLASMA PHYSICS FORMULAS AND DATA

PREFACE

The first edition of the NRL Plasma Formulary was compiled and published in 1974. By 1976 the original printing of 5000 copies had been exhausted. The time seemed ripe to prepare a revised edition incorporating corrections of errors reported by users. Extensive new material was also added (some of it already contained in the companion manual to the first edition, NRL Memorandum Report Number 2898), and a number of changes made in the interests of greater clarity and ease of use. It is hoped that the result represents a net improvement over its predecessor, and that the iterations will soon converge to a definitive version.

Two forms of the revised Formulary were prepared, a folding card like the 1974 edition, and a small booklet. With minor exceptions, they contain the same material. Their content is reproduced in the present memorandum report, which therefore serves as a less compact (but possibly more easily read) third version of the revised Formulary. It includes a list of errata for the folding card and booklet, complete as of the date of publication of the report. Where a formula in the pocket versions was incorrect, the corrected formula appears in this report marked with an asterisk. (Since the report itself was typed independently, however, it may contain errors not present in the pocket versions.)

In addition, the present report is intended to serve as a kind of users' manual for the Formulary, supplying explanations, worked examples, and references, for which space in the pocket versions was insufficient.

Manuscript submitted April 27, 1977.

A certain amount of supplementary material has been included, particularly in the Vector Calculus section. The elaborate reworking of the Atomic and Molecular Physics section included in the previous edition of this report has been abridged, however. Interested readers seeking background material on the formulas of this section are referred to the version originally prepared by A. W. Ali for Memorandum Report No. 2898 or to a forthcoming report, "Summary of Plasma Spectrographic Data," by J. Davis et. al., as well as to the numerous ponderous monographs on the subject.

Once again, many of my colleagues in NRL's Plasma Physics Division helped with the Formulary by suggesting changes and corrections and by proof reading. In addition to those persons previously acknowledged, particular thanks for specific contributions are due to Dr. Verne Jacobs of Science Applications, Inc. (Atomic Physics and Radiation), and Drs. Niels Winsor and Stephen Bodner of NRL (Collisions and Transport and Lasers, respectively). My special thanks go to Ms. Jean Monnette for managing to compose and lay out the Formulary while I was making continual changes in it.

ERRATA

March 1, 1977

(Foldout) Panel	(Booklet) Page	Erratum and Correction
2	4	Eq. (11): replace $\underline{B}\times\nabla\times\underline{A}$ by $\underline{B}\times(\nabla\times\underline{A})$
2	5	Eq. (17-19), (27): See p. 8
3	7	Under "Components of $(\underline{A}\cdot\underline{\nabla})\underline{B}$": replace $\frac{\partial B_z}{\partial_z}$ by $\frac{\partial B_z}{\partial z}$
4	10	Under "Rationalized mks" replace "mho" by "siemens"
5	11	Under "Rationalized mks" replace "weber/m^2" by "tesla"
5	12	Under "Rationalized mks" replace "newton/m^2" by "pascal"
8	34	Ion gyrofrequency, F region: replace 250 by 300
9	21	Def. of L: "horixontal" should be "horizontal"
14	33	In <u>Weakly Ionized Plasmas</u>, Eq. for collision frequency ν_α: T_α should be multiplied by k.
16	36	Eq. (5c): interchange 11.9 and 2.4; last Eq.: replace $A_1\sqrt{E}$ by $A_1/\sqrt{E}$
19	44	Eq. (15): should read $\alpha_3 = 8.75 \times 10^{-27} T_e^{-4.5}$ cm^6/sec

NUMERICAL AND ALGEBRAIC

Euler-Mascheroni constant[1] $\gamma = 0.57722$

Gamma Function $\Gamma(x + 1) = x\,\Gamma(x)$

$$\Gamma(\tfrac{1}{6}) = 5.5663$$

$$\Gamma(\tfrac{1}{5}) = 4.5908$$

$$\Gamma(\tfrac{1}{4}) = 3.6256$$

$$\Gamma(\tfrac{1}{3}) = 2.6789$$

$$\Gamma(\tfrac{2}{5}) = 2.2182$$

$$\Gamma(\tfrac{1}{2}) = 1.7725 = \pi^{\frac{1}{2}}$$

$$\Gamma(\tfrac{3}{5}) = 1.4892$$

$$\Gamma(\tfrac{2}{3}) = 1.3541$$

$$\Gamma(\tfrac{3}{4}) = 1.2254$$

$$\Gamma(\tfrac{4}{5}) = 1.1642$$

$$\Gamma(\tfrac{5}{6}) = 1.1288$$

$$\Gamma(1) = 1.0$$

Binomial Theorem (good for $|x| < 1$ or α = positive integer):

$$(1 + x)^{\alpha} = \sum_{k=0}^{\infty} \binom{\alpha}{k} x^k \cong 1 + \alpha x + \frac{\alpha(\alpha-1)}{2\,!} x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3\,!} x^3 + \ldots$$

Here

$$\binom{\alpha}{k} = \frac{\Gamma(\alpha+1)}{k!\,\Gamma(\alpha+1-k)} = \frac{\alpha(\alpha-1)\ldots(\alpha-k+1)}{k!}$$

is the binomial coefficient.

A very general relation involving the binomial coefficients is the Rothe-Hagen identity [2] (good for all complex x,y,z except when singular):

$$\sum_{k=0}^{n} \frac{x}{x+kz} \binom{x+kz}{k} \frac{y}{y+(n-k)z} \binom{y+(n-k)z}{n-k} = \frac{x+y}{x+y+nz} \binom{x+y+nz}{n} .$$

Many formulas involving binomial coefficients are special cases of the Rothe-Hagen identity. For example, with $z = 0$,

$$\sum_{k=0}^{n} \binom{x}{k} \binom{y}{n-k} = \binom{x+y}{n} ;$$

with $x = y = n, \quad z = 1$,

$$\sum_{k=0}^{n} \frac{(n+k-1)!}{k!} \frac{(2n-k-1)!}{(n-k)!} = \frac{2}{3} \frac{n!(3n)!}{n^2(2n)!} ;$$

with $x = - y = \frac{1}{2}, \quad z = \frac{1}{n}$, n odd,

$$\sum_{k=0}^{n} \frac{1}{n+2k} \frac{1}{n-2k} \binom{\frac{n+2k}{2n}}{k} \binom{\frac{n-2k}{2n}}{n-k} = 0 ,$$

etc.

Gain in decibels of P_2 relative to P_1

$$G = 10 \log_{10} (P_2/P_1) .$$

To within two percent

$$(2\pi)^{1/2} \approx 2.5; \quad e^3 \approx 20; \quad \pi^2 \approx 10; \quad 2^{10} \approx 10^3.$$

VECTOR CALCULUS

Notation: f, g, etc., are scalars; $\underline{A}$, $\underline{B}$, etc. are vectors; $\underline{\underline{T}}$ is a tensor

(1) $\underline{A}\cdot\underline{B} \times \underline{C} = \underline{A} \times \underline{B}\cdot\underline{C} = \underline{B}\cdot\underline{C} \times \underline{A} = \underline{B} \times \underline{C}\cdot\underline{A} = \underline{C}\cdot\underline{A} \times \underline{B} = \underline{C} \times \underline{A}\cdot\underline{B}$

(2) $\underline{A} \times (\underline{B} \times \underline{C}) = (\underline{A}\cdot\underline{C})\ \underline{B} - (\underline{A}\cdot\underline{B})\ \underline{C}$

(3) $\underline{A} \times (\underline{B} \times \underline{C}) + \underline{B} \times (\underline{C} \times \underline{A}) + \underline{C} \times (\underline{A} \times \underline{B}) = 0$

(4) $(\underline{A} \times \underline{B}) \cdot (\underline{C} \times \underline{D}) = (\underline{A}\cdot\underline{C})\ (\underline{B}\cdot\underline{D}) - (\underline{A}\cdot\underline{D})\ \ (\underline{B}\cdot\underline{C})$

(5) $(\underline{A} \times \underline{B}) \times (\underline{C} \times \underline{D}) = (\underline{A} \times \underline{B}\cdot\underline{D})\ \underline{C} - (\underline{A} \times \underline{B}\cdot\underline{C})\ \underline{D}$

(6) $\underline{\nabla}(fg) = \underline{\nabla}(gf) = f\,\underline{\nabla}\,g + g\,\underline{\nabla}\,f$

(7) $\underline{\nabla}\cdot(f\underline{A}) = f\,\underline{\nabla}\cdot\underline{A} + \underline{A}\cdot\underline{\nabla}\,f$

(8) $\underline{\nabla} \times (f\underline{A}) = f\,\underline{\nabla} \times \underline{A} + \underline{\nabla}f \times \underline{A}$

(9) $\underline{\nabla}\cdot(\underline{A} \times \underline{B}) = \underline{B}\cdot\underline{\nabla} \times \underline{A} - \underline{A}\cdot\underline{\nabla} \times \underline{B}$

(10) $\underline{\nabla} \times (\underline{A} \times \underline{B}) = \underline{A}\ (\underline{\nabla}\cdot\underline{B}) - \underline{B}\ (\underline{\nabla}\cdot\underline{A}) + (\underline{B}\cdot\underline{\nabla})\underline{A} - (\underline{A}\cdot\underline{\nabla})\ \underline{B}$

*(11) $\underline{\nabla}\ (\underline{A}\cdot\underline{B}) = \underline{A} \times (\underline{\nabla} \times \underline{B}) + \underline{B} \times (\underline{\nabla} \times \underline{A}) + (\underline{B}\cdot\underline{\nabla})\ \underline{A} + (\underline{A}\cdot\underline{\nabla})\ \underline{B}$

(12) $\nabla^2 f = \underline{\nabla}\cdot\underline{\nabla}\ f$

(13) $\underline{\nabla}^2\ \underline{A} = \underline{\nabla}(\underline{\nabla}\cdot\underline{A}) - \underline{\nabla} \times \underline{\nabla} \times \underline{A}$

(14) $\underline{\nabla} \times \underline{\nabla}f = 0$

(15) $\underline{\nabla}\cdot\underline{\nabla} \times \underline{A} = 0$

If $\underline{e}_1$, $\underline{e}_2$, $\underline{e}_3$ are orthonormal unit vectors, a second-order tensor $\underline{\underline{T}}$ can be written in the dyadic form

(16) $\underline{\underline{T}} = \sum_{i,j} T_{ij}\underline{e}_i\underline{e}_j$

In cartesian coordinates the divergence of a tensor is a vector with components given by

(17) $(\underline{\nabla} \cdot \underline{\underline{T}})_i = \sum_j (\partial T_{ji}/\partial x_j)$

In general

(18) $\underline{\nabla}\cdot(\underline{A}\underline{B}) = (\underline{\nabla} \cdot \underline{A})\underline{B} + (\underline{A} \cdot \underline{\nabla})\underline{B}$

(19) $\underline{\nabla}\cdot(f\underline{\underline{T}}) = \underline{\nabla}f \cdot \underline{\underline{T}} + f\,\underline{\nabla} \cdot \underline{\underline{T}}$

*Instead of (17), an alternative definition [4] was employed in the pocket versions of the Formulary:

$$(17') \quad (\underline{\nabla} \cdot \underline{\underline{T}})_i = \sum_j (\partial T_{ij}/\partial x_j)$$

Since we write the operator $\underline{\nabla}$ to the left, this choice, involving contraction with respect to the second index instead of the first, appears less "natural." When it is adopted, (18) and (19) become

$$(18') \quad \underline{\nabla} \cdot (\underline{A}\underline{B}) = \underline{A}\,\underline{\nabla} \cdot \underline{B} + \underline{B} \cdot \underline{\nabla}\underline{A}$$

$$(19') \quad \underline{\nabla} \cdot (f\underline{\underline{T}}) = \underline{\underline{T}} \cdot \underline{\nabla} f + f\underline{\nabla} \cdot \underline{\underline{T}}$$

But Gauss' theorem for tensors, formula (27), becomes

$$(27') \quad \int_V \underline{\nabla} \cdot \underline{\underline{T}}\, dV = \int_S \underline{\underline{T}} \cdot d\underline{S} \;,$$

which to the unsuspicious eye is equally "natural" looking. For symmetric tensors, the primed and unprimed formulas are identical. In plasma physics, fortunately, most of the tensors we deal with are symmetric. Nevertheless, the choice (17') is a potential source of confusion and its use in the other editions is regrettable. Here I have abandoned it. For uniformity, all of the integral formulas (25-34) have been written with differentials at the left, although this results in a significant change only in (27). The tensor divergence $\underline{\nabla} \cdot \underline{\underline{T}}$ in cylindrical and spherical coordinates (p. 12 and p. 15, respectively) now appears with indices transposed.

Let $\underline{r} = \underline{i}x + \underline{j}y + \underline{k}z$ be the radius vector of magnitude r, from the origin to the point x,y,z. Then

(20) $\underline{\nabla} \cdot \underline{r} = 3$

(21) $\underline{\nabla} \times \underline{r} = 0$

(22) $\underline{\nabla}\, r = \underline{r}/r$

(23) $\underline{\nabla}\,(1/r) = -\underline{r}/r^3$

(24) $\underline{\nabla} \cdot (\underline{r}/r^3) = 4\pi\delta(\underline{r})$

If V is a volume enclosed by a surface S and $d\underline{S} = \underline{n}\, dS$ where $\underline{n}$ is the unit normal outward from V,

(25) $$\int_V dV\, \underline{\nabla} f = \int_S d\underline{S}\, f$$

(26) $$\int_V dV\, \underline{\nabla} \cdot \underline{A} = \int_S d\underline{S} \cdot \underline{A}$$

(27) $$\int_V dV\, \underline{\nabla} \cdot \underline{\underline{T}} = \int_S d\underline{S} \cdot \underline{\underline{T}}$$

(28) $$\int_V dV\, \underline{\nabla} \times \underline{A} = \int_S d\underline{S} \times \underline{A}$$

(29) $$\int_V dV\, (f\, \underline{\nabla}^2 g - g \underline{\nabla}^2 f) = \int_S d\underline{S} \cdot (f\, \underline{\nabla} g - g\, \underline{\nabla} f)$$

(30) $$\int_V dV\, (\underline{A} \cdot \underline{\nabla} \times \underline{\nabla} \times \underline{B} \cdot \underline{\nabla} \times \underline{\nabla} \times \underline{A}) = \int_S d\underline{S} \cdot (\underline{B} \times \underline{\nabla} \times \underline{A} - \underline{A} \times \underline{\nabla} \times \underline{B})$$

If S is an open surface bounded by the contour C of which the line element is $d\underline{l}$,

(31) $\int_S d\underline{S} \times \underline{\nabla} f = \oint_C d\underline{l}\, f$

(32) $\int_S d\underline{S} \cdot \underline{\nabla} \times \underline{A} = \oint_C d\underline{l} \cdot \underline{A}$

(33) $\int_S (d\underline{S} \times \underline{\nabla}) \times \underline{A} = \oint_C d\underline{l} \times \underline{A}$

(34) $\int_S d\underline{S} \cdot (\underline{\nabla} f \times \underline{\nabla} g) = \oint_C f dg = - \oint_C g df$

Cylindrical Coordinates

Divergence

$$\underline{\nabla}\cdot\underline{A} = \frac{1}{r}\frac{\partial}{\partial r}(rA_r) + \frac{1}{r}\frac{\partial A_\theta}{\partial \theta} + \frac{\partial A_z}{\partial z}$$

Gradient

$$(\underline{\nabla} f)_r = \frac{\partial f}{\partial r}\ ; \quad (\underline{\nabla} f)_\theta = \frac{1}{r}\frac{\partial f}{\partial \theta}\ ; \quad (\underline{\nabla} f)_z = \frac{\partial f}{\partial z}\ ;$$

Curl

$$(\underline{\nabla} \times \underline{A})_r = \frac{1}{r}\frac{\partial A_z}{\partial \theta} - \frac{\partial A_\theta}{\partial z}$$

$$(\underline{\nabla} \times \underline{A})_\theta = \frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}$$

$$(\underline{\nabla} \times \underline{A})_z = \frac{1}{r}\frac{\partial}{\partial r}(rA_\theta) - \frac{1}{r}\frac{\partial A_r}{\partial \theta}$$

Laplacian

$$\nabla^2 f = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2} + \frac{\partial^2 f}{\partial z^2}$$

Laplacian of a vector

$$(\nabla^2 \underline{A})_r = \nabla^2 A_r - \frac{2}{r^2}\frac{\partial A_\theta}{\partial \theta} - \frac{A_r}{r^2}$$

$$(\nabla^2 \underline{A})_\theta = \nabla^2 A_\theta + \frac{2}{r^2}\frac{\partial A_r}{\partial \theta} - \frac{A_\theta}{r^2}$$

$$(\nabla^2 \underline{A})_z = \nabla^2 A_z$$

Components of $(\underline{A}\cdot\underline{\nabla})\underline{B}$

$$(\underline{A}\cdot\underline{\nabla}\underline{B})_r = A_r \frac{\partial B_r}{\partial r} + \frac{A_\theta}{r}\frac{\partial B_r}{\partial \theta} + A_z \frac{\partial B_r}{\partial z} - \frac{A_\theta B_\theta}{r}$$

$$(\underline{A}\cdot\underline{\nabla}\underline{B})_\theta = A_r \frac{\partial B_\theta}{\partial r} + \frac{A_\theta}{r}\frac{\partial B_\theta}{\partial \theta} + A_z \frac{\partial B_\theta}{\partial z} + \frac{A_\theta B_r}{r}$$

$$^*(\underline{A}\cdot\underline{\nabla}\underline{B})_z = A_r \frac{\partial B_z}{\partial r} + \frac{A_\theta}{r}\frac{\partial B_z}{\partial \theta} + A_z \frac{\partial B_z}{\partial z}$$

Divergence of a tensor

$$(\underline{\nabla}\cdot\underline{\underline{T}})_r = \frac{1}{r}\frac{\partial}{\partial r}(rT_{rr}) + \frac{1}{r}\frac{\partial}{\partial \theta}(T_{\theta r}) + \frac{\partial T_{zr}}{\partial z} - \frac{1}{r}T_{\theta\theta}$$

$$(\underline{\nabla}\cdot\underline{\underline{T}})_\theta = \frac{1}{r}\frac{\partial}{\partial r}(rT_{r\theta}) + \frac{1}{r}\frac{\partial T_{\theta\theta}}{\partial \theta} + \frac{\partial T_{z\theta}}{\partial z} + \frac{1}{r}T_{\theta r}$$

$$(\underline{\nabla}\cdot\underline{\underline{T}})_z = \frac{1}{r}\frac{\partial}{\partial r}(rT_{rz}) + \frac{1}{r}\frac{\partial T_{\theta z}}{\partial \theta} + \frac{\partial T_{zz}}{\partial z}$$

Spherical Coordinates

Divergence

$$\nabla \cdot A = \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 A_r) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (A_\theta \sin \theta) + \frac{1}{r \sin \theta} \frac{\partial A_\phi}{\partial \phi}$$

Gradient

$$(\nabla f)_r = \frac{\partial f}{\partial r} ; \quad (\nabla f)_\theta = \frac{1}{r} \frac{\partial f}{\partial \theta} ; \quad (\nabla f)_\phi = \frac{1}{r \sin \theta} \frac{\partial f}{\partial \phi}$$

Curl

$$(\nabla \times A)_r = \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (A_\phi \sin \theta) - \frac{1}{r \sin \theta} \frac{\partial A_\theta}{\partial \phi}$$

$$(\nabla \times A)_\theta = \frac{1}{r \sin \theta} \frac{\partial A_r}{\partial \phi} - \frac{1}{r} \frac{\partial}{\partial r} (r A_\phi)$$

$$(\nabla \times A)_\phi = \frac{1}{r} \frac{\partial}{\partial r} (r A_\theta) - \frac{1}{r} \frac{\partial A_r}{\partial \theta}$$

Laplacian

$$\nabla^2 f = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial f}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial f}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 f}{\partial \phi^2}$$

Laplacian of a vector

$$(\nabla^2 \underline{A})_r = \nabla^2 A_r - \frac{2A_r}{r^2} - \frac{2}{r^2}\frac{\partial A_\theta}{\partial\theta} - \frac{2A_\theta \cot\theta}{r^2} - \frac{2}{r^2 \sin\theta}\frac{\partial A_\phi}{\partial\phi}$$

$$(\nabla^2 \underline{A})_\theta = \nabla^2 A_\theta + \frac{2}{r^2}\frac{\partial A_r}{\partial\theta} - \frac{A_\theta}{r^2 \sin^2\theta} - \frac{2\cos\theta}{r^2 \sin^2\theta}\frac{\partial A_\phi}{\partial\phi}$$

$$(\nabla^2 \underline{A})_\phi = \nabla^2 A_\phi - \frac{A_\phi}{r^2 \sin^2\theta} + \frac{2}{r^2 \sin\theta}\frac{\partial A_r}{\partial\phi} + \frac{2\cos\theta}{r^2 \sin^2\theta}\frac{\partial A_\theta}{\partial\phi}$$

Components of $(\underline{A}\cdot\underline{\nabla})\underline{B}$

$$(\underline{A}\cdot\underline{\nabla}\underline{B})_r = A_r\frac{\partial B_r}{\partial r} + \frac{A_\theta}{r}\frac{\partial B_r}{\partial\theta} + \frac{A_\phi}{r\sin\theta}\frac{\partial B_r}{\partial\phi} - \frac{A_\theta B_\theta + A_\phi B_\phi}{r}$$

$$(\underline{A}\cdot\underline{\nabla}\underline{B})_\theta = A_r\frac{\partial B_\theta}{\partial r} + \frac{A_\theta}{r}\frac{\partial B_\theta}{\partial\theta} + \frac{A_\phi}{r\sin\theta}\frac{\partial B_\theta}{\partial\phi} + \frac{A_\theta B_r}{r} - \frac{A_\phi B_\phi \cot\theta}{r}$$

$$(\underline{A}\cdot\underline{\nabla}\underline{B})_\phi = A_r\frac{\partial B_\phi}{\partial r} + \frac{A_\theta}{r}\frac{\partial B_\phi}{\partial\theta} + \frac{A_\phi}{r\sin\theta}\frac{\partial B_\phi}{\partial\phi} + \frac{A_\phi B_r}{r} + \frac{A_\phi B_\theta \cot\theta}{r}$$

Divergence of a tensor

$$(\underline{\nabla}\cdot\underline{\underline{T}})_r = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 T_{rr}) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(T_{\theta r}\sin\theta)$$

$$+ \frac{1}{r\sin\theta}\frac{\partial T_{\phi r}}{\partial\phi} - \frac{T_{\theta\theta} + T_{\phi\phi}}{r}$$

$$(\underline{\nabla}\cdot\underline{\underline{T}})_\theta = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 T_{r\theta}) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(T_{\theta\theta}\sin\theta)$$

$$+ \frac{1}{r\sin\theta}\frac{\partial T_{\phi\theta}}{\partial\phi} + \frac{T_{\theta r}}{r} - \frac{\cot\theta}{r}T_{\phi\phi}$$

$$(\underline{\nabla}\cdot\underline{\underline{T}})_\phi = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 T_{r\phi}) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(T_{\theta\phi}\sin\theta) + \frac{1}{r\sin\theta}\frac{\partial T_{\phi\phi}}{\partial\phi} + \frac{T_{\phi r}}{r}$$

$$+ \frac{\cot\theta}{r}T_{\phi\theta}$$

Note that the expressions given for $\underline{\nabla}\cdot\underline{\underline{T}}$ are consistent with the definition (17).

General Curvilinear Coordinates [3,4]

In any orthogonal curvilinear coordinate system, all of the operations involving $\underline{\nabla}$ can be defined in terms of the scale factors h_i. These are related to the differential arc length $d\ell$ by

$$
\begin{aligned}
d\ell^2 &= dx^2 + dy^2 + dz^2 \\
&= h_1^2 d\xi_1^2 + h_2^2 d\xi_2^2 + h_3^2 d\xi_3^2 \ ,
\end{aligned} \tag{35}
$$

where $\{\xi_1, \xi_2, \xi_3\}$ designate the curvilinear coordinates. For example, the scale factors associated with the spherical coordinates $\{r, \theta, \varphi\}$ are 1, r and r sin θ, respectively.

In orthogonal curvilinear coordinate systems a vector $\underline{A}$ can be written as

$$\underline{A} = \underline{e}_1 A_1 + \underline{e}_2 A_2 + \underline{e}_3 A_3 \ , \tag{36}$$

where the $\{\underline{e}_i\}$ form a right-handed set of unit vectors normal to the surfaces of constant ξ_i. Then the divergence of $\underline{A}$ is

$$\underline{\nabla} \cdot \underline{A} = \frac{1}{H} \sum_i \frac{\partial}{\partial \xi_i} \left(\frac{HA_i}{h_i} \right) \ , \tag{37}$$

where $H = h_1 h_2 h_3$. The gradient of a scalar f is given by

$$(\underline{\nabla} f)_i = \frac{1}{h_i} \frac{\partial f}{\partial \xi_i} \ . \tag{38}$$

From Eqs. (12), (37) and (38), it follows that the Laplacian of f is

(39) $$\nabla^2 f = \frac{1}{H} \sum_i \frac{\partial}{\partial \xi_i} \left(\frac{H}{h_i^2} \frac{\partial f}{\partial \xi_i} \right) .$$

The curl of $\underline{A}$ is

(40) $$(\underline{\nabla} \times \underline{A})_i = \sum_{j,k} \epsilon_{ijk} \frac{1}{h_j h_k} \frac{\partial}{\partial \xi_j} (h_k A_k) ,$$

where ϵ_{ijk} equals + 1 if $\{ijk\}$ is even (i.e., an even permutation of $\{123\}$), equals - 1 if $\{ijk\}$ is odd, and vanishes if i,j and k are not all distinct.

To obtain expressions for the remaining operations, it is useful to introduce the Christoffel symbols Γ^i_{jk}, defined by

(41) $$\Gamma^i_{jk} = \frac{1}{h_j h_k} \left[\frac{\partial h_j}{\partial \xi_k} \delta^i_j - \frac{\partial h_k}{\partial \xi_j} \delta^i_k \right] ,$$

where δ^i_j equals 1 if i = j and vanishes otherwise. Γ^i_{jk} vanishes if j and k both differ from i or if j = k. In addition,

(42) $$\Gamma^i_{jk} + \Gamma^i_{kj} = 0 ;$$

(43) $$\Gamma^i_{jk} + \Gamma^j_{ki} + \Gamma^k_{ij} = 0 ;$$

and

(44) $$\sum_i \Gamma^i_{ik} = \frac{1}{H} \frac{\partial}{\partial \xi_k} \left(\frac{H}{h_k} \right) .$$

There are at most six independent non-zero components Γ^i_{jk}, and less in most coordinate systems of physical interest. The curvature tensor,

$$R_{ijk\ell} = \frac{1}{h_j}\frac{\partial \Gamma^i_{k\ell}}{\partial \xi_j} - \frac{1}{h_i}\frac{\partial \Gamma^j_{k\ell}}{\partial \xi_i} + \sum_n \left\{ \Gamma^n_{ij}\Gamma^n_{k\ell} - \Gamma^j_{nk}\Gamma^i_{n\ell} + \Gamma^i_{kn}\Gamma^j_{\ell n} \right\} \tag{45}$$

always vanishes in a flat space, i.e., one which can be described by Cartesian coordinates. (This is the only kind of space we usually deal with in plasma physics.)

In terms of the Γ's, the gradient of a vector $\underline{A}$ is a tensor $\underline{\underline{T}}$ given by

$$T_{ji} = (\underline{\nabla}\,\underline{A})_{ji} = \frac{\partial A_i}{h_j \partial \xi_j} + \sum_\ell \Gamma^j_{i\ell} A_\ell \; . \tag{46}$$

Contracting this expression (i.e., multiplying by δ^i_j and summing over i and j) recovers Eq. (37). The directional derivative is

$$\begin{aligned} \left[(\underline{A}\cdot\underline{\nabla})\underline{B}\right]_i &= \sum_j A_j (\underline{\nabla}\,\underline{B})_{ji} \\ &= \sum_j \left\{ \frac{A_j}{h_j}\frac{\partial B_i}{\partial \xi_j} + \frac{A_i B_j}{h_i h_j}\frac{\partial h_i}{\partial \xi_j} - \frac{A_j B_j}{h_i h_j}\frac{\partial h_j}{\partial \xi_i} \right\} \; . \end{aligned} \tag{47}$$

The gradient of a (second-order) tensor is a third-order tensor given by

(48) $$(\nabla \underline{\underline{T}})_{kji} = \frac{\partial T_{ji}}{h_k \partial \xi_k} + \sum_{\ell} \left\{ \Gamma^k_{i\ell} T_{j\ell} + \Gamma^k_{j\ell} T_{\ell i} \right\} .$$

The divergence of the tensor $\underline{\underline{T}}$ is obtained by contracting Eq. (48) with respect to the first two indices:

$$(\nabla \cdot \underline{\underline{T}})_i = \sum_j \left\{ \frac{\partial T_{ji}}{h_j \partial \xi_j} + \sum_{\ell} \left[\Gamma^j_{i\ell} T_{j\ell} + \Gamma^j_{j\ell} T_{\ell i} \right] \right\}$$

$$= \sum_j \left\{ \frac{1}{H} \frac{\partial}{\partial \xi_j} \left[\frac{H T_{ji}}{h_j} \right] + \sum_{\ell} \Gamma^j_{i\ell} T_{j\ell} \right\} .$$

If $\underline{\underline{T}}$ is defined according to Eq. (46), $\nabla \cdot \underline{\underline{T}}$ is the Laplacian of the vector $\underline{A}$:

$$(\nabla^2 \underline{A})_i = \sum_j (\nabla \nabla \underline{A})_{jji} = \sum_j \left\{ \frac{1}{H} \frac{\partial}{\partial \xi_j} \left[\frac{H}{h_j^2} \frac{\partial A_i}{\partial \xi_j} \right] \right.$$

(50) $$+ \frac{h_i A_i}{H} \frac{\partial}{\partial \xi_j} \left[\frac{H}{h_i^2 h_j^2} \frac{\partial h_i}{\partial \xi_j} \right] + \frac{2}{h_i^2 h_j^2} \frac{\partial h_i}{\partial \xi_j} \frac{\partial}{\partial \xi_i} (h_j A_j)$$

$$\left. - \frac{2}{h_i h_j^3} \frac{\partial h_j}{\partial \xi_i} \frac{\partial}{\partial \xi_j} (h_j A_j) + \frac{h_j A_j}{h_i} \frac{\partial}{\partial \xi_i} \left[\frac{1}{H} \frac{\partial}{\partial \xi_j} \left(\frac{H}{h_j^2} \right) \right] \right\} .$$

Evidently the first term on the RHS is $\nabla^2 A_i$, the scalar Laplacian applied to A_i [Eq. (39)]. Fortunately, Eq. (50) simplifies considerably for most coordinate systems of interest.

In the following section, the results of Eqs. (35) - (50) are applied to a coordinate system occasionally encountered in plasma physics problems.

Toroidal Coordinates

An arbitrary point in 3-space can be uniquely identified by the coordinates $\{\rho, \phi, \psi\}$, where ρ is the distance between the given point and a fixed toroidal axis of radius R (the major radius), ϕ is the azimuthal or toroidal angle (i.e., the one sweeping around the major circumference), and ψ is the poloidal angle. (Note that this system differs from the "toroidal" coordinates in Ref. 3, p. 666, and from the flux surface coordinates commonly used in connection with toroidal devices.) The values assumed by these coordinates are restricted according to $0 \leq \rho < \infty$, $0 \leq \phi < 2\pi$, and $0 \leq \psi < 2\pi$. If $\{r, \phi, z\}$ are the cylindrical coordinates of the point in question, where z is normal to the plane of the torus, then ϕ is the same in both systems and

$$R + \rho \cos \psi = r,$$
$$\rho \sin \psi = z .$$

Hence the square of the differential are length is

$$d\ell^2 = dr^2 + r^2 d\phi^2 + dz^2$$
$$= d\rho^2 + (R + \rho \cos \psi)^2 d\phi^2 + \rho^2 d\psi^2 .$$

Consequently the scale factors are 1, $R + \rho \cos \psi$, and ρ, respectively.

From Eq. (37), the divergence of $\underset{\sim}{A}$ is

$$\underline{\nabla}\cdot\underline{A} = \frac{1}{R + \rho \cos \psi} \left\{ \frac{1}{\rho} \frac{\partial}{\partial \rho} \left[\rho(R + \rho \cos \psi) A_\rho \right] + \right.$$

$$\left. \frac{\partial A_\phi}{\partial \phi} + \frac{1}{\rho} \frac{\partial}{\partial \psi} \left[(R + \rho \cos \psi) A_\psi \right] \right\} .$$

From Eq. (38), the gradient of f has components

$$(\nabla f)_\rho = \frac{\partial f}{\partial \rho} ;$$

$$(\nabla f)_\phi = \frac{1}{R + \rho \cos \psi} \frac{\partial f}{\partial \phi} ;$$

$$(\nabla f)_\psi = \frac{1}{\rho} \frac{\partial f}{\partial \psi} .$$

From Eq. (39), the Laplacian of f is

$$\nabla^2 f = \frac{1}{R + \rho \cos \psi} \left\{ \frac{1}{\rho} \frac{\partial}{\partial \rho} \left[\rho(R + \rho \cos \psi) \frac{\partial f}{\partial \rho} \right] \right.$$

$$\left. + \frac{1}{R + \rho \cos \psi} \frac{\partial^2 f}{\partial \phi^2} + \frac{1}{\rho^2} \frac{\partial}{\partial \psi} \left[(R + \rho \cos \psi) \frac{\partial f}{\partial \psi} \right] \right\} .$$

From Eq. (40), the curl of $\underline{A}$ is given by

$$(\underline{\nabla} \times \underline{A})_\rho = \frac{1}{R + \rho \cos \psi} \left\{ \frac{\partial A_\psi}{\partial \phi} - \frac{1}{\rho} \frac{\partial}{\partial \psi} \left[(R + \rho \cos \psi) A_\phi \right] \right\} ;$$

$$(\underline{\nabla} \times \underline{A})_\phi = \frac{1}{\rho} \left\{ \frac{\partial A_\rho}{\partial \psi} - \frac{\partial}{\partial \rho} (\rho A_\psi) \right\} ;$$

$$(\underline{\nabla} \times \underline{A})_\psi = \frac{1}{R + \rho \cos \psi} \left\{ \frac{\partial}{\partial \rho} \left[(R + \rho \cos \psi) A_\phi - \frac{\partial A_\rho}{\partial \phi} \right] \right\} .$$

The only nonvanishing Christoffel symbols are

$$\Gamma^2_{21} = - \Gamma^2_{12} = \frac{\cos \psi}{R + \rho \cos \psi} ;$$

$$\Gamma^2_{23} = - \Gamma^2_{32} = \frac{- \sin \psi}{R + \rho \cos \psi} ;$$

$$\Gamma^3_{31} = - \Gamma^3_{13} = \frac{1}{\rho} .$$

Then from Eq. (47),

$$\left[(\underline{A} \cdot \underline{\nabla}) \underline{B} \right]_\rho = A_\rho \frac{\partial B_\rho}{\partial \rho} + \frac{A_\phi}{R + \rho \cos \psi} \frac{\partial B_\rho}{\partial \phi}$$

$$+ \frac{A_\psi}{\rho} \frac{\partial B_\rho}{\partial \psi} - \frac{A_\phi B_\phi \cos \psi}{R + \rho \cos \psi} - \frac{A_\psi B_\psi}{\rho} ;$$

$$\left[(\underline{A}\cdot\underline{\nabla})\underline{B}\right]_\phi = A_\rho \frac{\partial B_\phi}{\partial\rho} + \frac{A_\phi}{R+\rho\cos\psi}\frac{\partial B_\phi}{\partial\phi} +$$

$$\frac{A_\psi}{\rho}\frac{\partial B_\phi}{\partial\psi} + \frac{A_\phi B_\rho \cos\psi}{R+\rho\cos\psi} - \frac{A_\phi B_\psi \sin\psi}{R+\rho\cos\psi} \quad ;$$

$$\left[(\underline{A}\cdot\underline{\nabla})\underline{B}\right]_\psi = A_\rho \frac{\partial B_\psi}{\partial\rho} + \frac{A_\phi}{R+\rho\cos\psi}\frac{\partial B_\psi}{\partial\phi} +$$

$$\frac{A_\psi}{\rho}\frac{\partial B_\psi}{\partial\psi} + \frac{A_\psi B_\rho}{\rho} + \frac{A_\phi B_\phi \sin\psi}{R+\rho\cos\psi} \quad .$$

From Eq. (49),

$$(\underline{\nabla}\cdot\underline{\underline{T}})_\rho = \frac{1}{R+\rho\cos\psi}\left\{\frac{1}{\rho}\frac{\partial}{\partial\rho}\left[\rho(R+\rho\cos\psi)T_{\rho\rho}\right]\right.$$

$$+\frac{\partial T_{\phi\rho}}{\partial\phi} + \frac{1}{\rho}\frac{\partial}{\partial\psi}\left[(R+\rho\cos\psi)T_{\psi\rho}\right]$$

$$\left. - \cos\psi T_{\phi\phi}\right\} - \frac{1}{\rho}T_{\psi\psi} \quad ;$$

$$(\underline{\nabla} \cdot \underline{\underline{T}})_{\phi} = \frac{1}{R + \rho \cos \psi} \left\{ \frac{1}{\rho} \frac{\partial}{\partial \rho} \left[\rho(R + \rho \cos \psi) T_{\rho\phi} \right] \right.$$

$$+ \frac{\partial T_{\phi\phi}}{\partial \phi} + \frac{1}{\rho} \frac{\partial}{\partial \psi} \left[(R + \rho \cos \psi) T_{\psi\phi} \right]$$

$$\left. + \cos \psi T_{\phi\rho} - \sin \psi T_{\phi\psi} \right\} ;$$

$$(\underline{\nabla} \cdot \underline{\underline{T}})_{\psi} = \frac{1}{R + \rho \cos \psi} \left\{ \frac{1}{\rho} \frac{\partial}{\partial \rho} \left[\rho(R + \rho \cos \psi) T_{\rho\psi} \right] \right.$$

$$+ \frac{\partial T_{\phi\psi}}{\partial \phi} + \frac{1}{\rho} \frac{\partial}{\partial \psi} \left[(R + \rho \cos \psi) T_{\psi\psi} \right]$$

$$\left. + \sin \psi T_{\phi\phi} \right\} + \frac{1}{\rho} T_{\psi\rho} \quad .$$

Finally, from Eq. (50),

$$(\nabla^2 \underline{A})_{\rho} = \nabla^2 A_{\rho} - A_{\rho} \left[\frac{1}{\rho^2} + \frac{\cos^2 \psi}{(R + \rho \cos \psi)^2} \right]$$

$$- \frac{2 \cos \psi}{(R + \rho \cos \psi)^2} \frac{\partial A_{\phi}}{\partial \phi} - \frac{2}{\rho^2} \frac{\partial A_{\psi}}{\partial \psi} +$$

$$\frac{\sin \psi \, A_{\psi}}{R + \rho \cos \psi} \left[\frac{1}{\rho} + \frac{\cos \psi}{R + \rho \cos \psi} \right] ;$$

$$(\nabla^2 \underline{A})_\phi = \nabla^2 A_\phi - \frac{1}{(R + \rho \cos \psi)^2} A_\phi$$

$$+ \frac{2}{(R + \rho \cos \psi)^2} \left[\cos \psi \frac{\partial A_\rho}{\partial \phi} - \sin \psi \frac{\partial A_\psi}{\partial \phi} \right] ;$$

$$(\nabla^2 \underline{A})_\psi = \nabla^2 A_\psi - A_\psi \left[\frac{1}{\rho^2} + \frac{\sin^2 \psi}{(R + \rho \cos \psi)^2} \right]$$

$$+ \frac{2}{\rho^2} \frac{\partial A_\rho}{\partial \psi} + \frac{\sin \psi}{(R + \rho \cos \psi)^2} \left[2 \frac{\partial A_\phi}{\partial \phi} - \frac{RA_\rho}{\rho} \right] .$$

DIMENSIONS AND UNITS [5]

To get the value of a quantity in Gaussian units, multiply the value expressed in mks units by the conversion factor.

Physical Quantity	Symbol	Dimensions: mks Units	Dimensions: Gaussian Units	Rationalized mks	Conversion Factor	Gaussian
Capacitance	C	$\frac{t^2q^2}{m\ell^2}$	ℓ	farad	9×10^{11}	cm
Charge	q	q	$\frac{m^{1/2}\ell^{3/2}}{t}$	coulomb	3×10^9	statcoulomb
Charge density	ρ	$\frac{q}{\ell^3}$	$\frac{m^{1/2}}{\ell^{3/2}t}$	coulomb/m^3	3×10^3	statcoulomb/cm^3
Conductance		$\frac{tq^2}{m\ell^2}$	$\frac{\ell}{t}$	siemens*	9×10^{11}	cm/sec
Conductivity	σ	$\frac{tq^2}{m\ell^3}$	$\frac{1}{t}$	siemens/m*	9×10^9	sec^{-1}
Current	I	$\frac{q}{t}$	$\frac{m^{1/2}\ell^{3/2}}{t^2}$	ampere	3×10^9	statampere
Current density	$\underline{J}$	$\frac{q}{\ell^2 t}$	$\frac{m^{1/2}}{\ell^{1/2}t^2}$	ampere/m^2	3×10^5	statampere/cm^2
Density	ρ	$\frac{m}{\ell^3}$	$\frac{m}{\ell^3}$	kg/m^3	10^{-3}	g/cm^3

Physical Quantity	Symbol	Dimensions: mks Units	Dimensions: Gaussian Units	Rationalized mks	Conversion Factor	Gaussian
Displacement	$\underline{D}$	$\frac{q}{\ell^2}$	$\frac{m^{1/2}}{\ell^{1/2}t}$	coulomb/m^2	$12\pi \times 10^5$	statcoulomb/cm^2
Electric field	$\underline{E}$	$\frac{m\ell}{t^2q}$	$\frac{m^{1/2}}{t\ell^{1/2}}$	volt/m	$\frac{1}{3} \times 10^{-4}$	statvolt/cm
Electromotance	$\mathcal{E}$, Emf	$\frac{m\ell^2}{t^2q}$	$\frac{m^{1/2}\ell^{1/2}}{t}$	volt	$\frac{1}{3} \times 10^{-2}$	statvolt
Energy	U, W	$\frac{m\ell^2}{t^2}$	$\frac{m\ell^2}{t^2}$	joule	10^7	erg
Energy density		$\frac{m}{\ell t^2}$	$\frac{m}{\ell t^2}$	joule/m^3	10	erg/cm^3
Force	$\underline{F}$	$\frac{m\ell}{t^2}$	$\frac{m\ell}{t^2}$	newton	10^5	dyne
Frequency	f, ν, ω	$\frac{1}{t}$	$\frac{1}{t}$	hertz	1	hertz

Physical Quantity	Symbol	Dimensions: mks Units	Dimensions: Gaussian Units	Rationalized mks	Conversion Factor	Gaussian
Impedance	Z	$\frac{m\ell^2}{tq^2}$	$\frac{t}{\ell}$	ohm	$\frac{1}{9} \times 10^{-11}$	sec/cm
Inductance	L	$\frac{m\ell^2}{q^2}$	$\frac{t^2}{\ell}$	henry	$\frac{1}{9} \times 10^{-11}$	sec^2/cm
Length	ℓ	ℓ	ℓ	meter (m)	10^2	centimeter (cm)
Magnetic intensity	$\underline{H}$	$\frac{q}{\ell t}$	$\frac{m^{1/2}}{\ell^{1/2}\, t}$	ampere-turn/m	$4\pi \times 10^{-3}$	oersted
Magnetic flux	Φ	$\frac{m\ell^2}{tq}$	$\frac{m^{1/2}\ell^{3/2}}{t}$	weber	10^8	maxwell
Magnetic induction	$\underline{B}$	$\frac{m}{tq}$	$\frac{m^{1/2}}{\ell^{1/2}\, t}$	tesla *	10^4	gauss
Magnetization	$\underline{M}$	$\frac{q}{\ell t}$	$\frac{m^{1/2}}{\ell^{1/2}\, t}$	ampere-turn/m	10^{-3}	oersted
Magnetomotance	Mmf, $\mathcal{M}$	$\frac{q}{t}$	$\frac{m^{1/2}\,\ell^{1/2}}{t}$	ampere-turn	$\frac{4\pi}{10}$	gilbert
Mass	m, M	m	m	kilogram (kg)	10^3	gram (g)
Momentum	$\underline{p}$, $\underline{P}$	$\frac{m\ell}{t}$	$\frac{m\ell}{t}$	kg-m/sec	10^5	g-cm/sec

Physical Quantity	Symbol	Dimensions: mks Units	Dimensions: Gaussian Units	Rationalized mks	Conversion Factor	Gaussian
Momentum density		$\frac{m}{\ell^2 t}$	$\frac{m}{\ell^2 t}$	kg/m^2-sec	10^{-1}	g/cm^2-sec
Permeability	μ	$\frac{m\ell}{q^2}$	1	henry/m	$\frac{1}{4\pi} \times 10^7$	(dimensionless)
Permittivity	ϵ	$\frac{t^2 q^2}{m\ell^3}$	1	farad/m	$36\pi \times 10^9$	(dimensionless)
Polarization	$\underline{P}$	$\frac{q}{\ell^2}$	$\frac{m^{1/2}}{\ell^{1/2} t}$	coulomb/m^2	3×10^5	statcoulomb/cm^2
Potential	V, ϕ	$\frac{m\ell^2}{t^2 q}$	$\frac{m^{1/2}\ell^{1/2}}{t}$	volt	$\frac{1}{3} \times 10^{-2}$	statvolt
Power	P	$\frac{m\ell^2}{t^3}$	$\frac{m\ell^2}{t^3}$	watt	10^7	erg/sec
Power density		$\frac{m}{\ell t^3}$	$\frac{m}{\ell t^3}$	watt/m^3	10	erg/cm^3-sec
Pressure	p	$\frac{m}{\ell t^2}$	$\frac{m}{\ell t^2}$	pascal*	10	dyne/cm^2
Reluctance	$\mathcal{R}$	$\frac{q^2}{m\ell^2}$	$\frac{1}{\ell}$	ampere-turn/weber	$4\pi \times 10^{-9}$	cm^{-1}

Physical Quantity	Symbol	Dimensions: mks Units	Dimensions: Gaussian Units	Rationalized mks	Conversion Factor	Gaussian
Resistance	R	$\frac{m\ell^2}{tq^2}$	$\frac{t}{\ell}$	ohm	$\frac{1}{9} \times 10^{-11}$	sec/cm
Resistivity	η, ρ	$\frac{m\ell^3}{tq^2}$	t	ohm-m	$\frac{1}{9} \times 10^{-9}$	sec
Thermal conductivity	κ	$\frac{m\ell}{t^3}$	$\frac{m\ell}{t^3}$	watt/m-deg(K)	10^5	erg/cm-sec-deg(K)
Time	t	t	t	second(sec)	1	second
Vector potential	$\underline{A}$	$\frac{m\ell}{tq}$	$\frac{m^{1/2}\ell^{1/2}}{t}$	weber/m	10^6	gauss-cm
Velocity	$\underline{v}$	$\frac{\ell}{t}$	$\frac{\ell}{t}$	m/sec	10^2	cm/sec
Viscosity	η, μ	$\frac{m}{\ell t}$	$\frac{m}{\ell t}$	kg/m-sec	10	poise
Vorticity	ζ	$\frac{1}{t}$	$\frac{1}{t}$	sec^{-1}	1	sec^{-1}
Work	W	$\frac{m\ell^2}{t^2}$	$\frac{m\ell^2}{t^2}$	joule	10^7	erg

PHYSICAL CONSTANTS [5,6]

cgs

Physical Quantity	Symbol	Value	Units
Boltzmann constant	k	1.3807×10^{-16}	erg/deg (K)
Elementary charge	e	4.8032×10^{-10}	statcoulomb
Electron mass	m_e	9.1095×10^{-28}	g
Proton mass	m_p	1.6726×10^{-24}	g
Gravitational constant	G	6.6720×10^{-8}	dyne-cm^2/g^2
Planck constant	h	6.6262×10^{-27}	erg-sec
	$\hbar = h/2\pi$	1.0546×10^{-27}	erg-sec
Speed of light in vacuum	c	2.9979×10^{10}	cm/sec

Physical Quantity	Symbol	Value	Units
Proton/electron mass ratio	m_p/m_e	1.8362×10^{3}	
Electron charge/mass ratio	e/m_e	5.2728×10^{17}	statcoulomb/g
Rydberg constant	$R_\infty = \frac{2\pi^2 me^4}{ch^3}$	1.0974×10^{5}	cm^{-1}
Bohr radius	$a_o = \frac{\hbar^2}{me^2}$	5.2918×10^{-9}	cm
Atomic cross section	πa_o^2	8.7974×10^{-17}	cm^2
Classical electron radius	$r_e = e^2/mc^2$	2.8179×10^{-13}	cm
Thomson cross section	$(8\pi/3)r_e^2$	6.6524×10^{-25}	cm^2
Compton wavelength of electron	$h/m_e c$	2.4263×10^{-10}	cm
	$\hbar/m_e c$	3.8616×10^{-11}	cm
Fine-structure constant	$\alpha = \frac{e^2}{\hbar c}$	7.2974×10^{-3}	
	α^{-1}	137.04	

Physical Quantity	Symbol	Value	Units
First radiation constant	$c_1 = 2\pi hc^2$	3.7418×10^{-5}	erg-cm^2/sec
Second radiation constant	$c_2 = hc/k$	1.4388	cm-deg(K)
Stefan-Boltzmann constant	σ	5.6703×10^{-5}	erg/cm^2-sec-deg^4
Wavelength associated with 1 eV	λ_o	1.2399×10^{-4}	cm
Frequency associated with 1 eV	ν_o	2.4180×10^{14}	Hz
Wave number associated with 1 eV	k_o	8.0655×10^{3}	cm^{-1}
Energy associated with 1 eV		1.6022×10^{-12}	erg
Energy associated with 1 cm^{-1}		1.9865×10^{-16}	erg
Energy associated with 1 Rydberg		13.606	eV
Energy associated with 1^o Kelvin		8.6173×10^{-5}	eV
Temperature associated with 1 eV		1.1605×10^{4}	deg(K)
Avogadro number	N_A	6.0220×10^{23}	mol^{-1}

Physical Quantity	Symbol	Value	Units
Faraday constant	$F = N_A e$	2.8925×10^{14}	statcoulomb/mol
Gas constant	$R = N_A k$	8.3144×10^{7}	erg/deg-mol
Loschmidt's number (no. density at STP)	n_o	2.6868×10^{19}	cm^{-3}
Atomic mass unit	m_u	1.6606×10^{-24}	g
Standard temperature	T_o	273.16	deg (K)
Atmospheric pressure	$p_o = n_o k T_o$	1.0133×10^{6}	$dyne/cm^2$
Pressure of 1 mm Hg(torr)		1.3332×10^{3}	$dyne/cm^2$
Molar volume at STP	$V_o = RT_o/p_o$	2.2415×10^{4}	cm^3
calorie (cal)		4.1868×10^{7}	erg
Gravitational acceleration	g	980.67	cm/sec^2

METRIC PREFIXES [5]

Multiple	Prefix	Symbol	Multiple	Prefix	Symbol
10^{-1}	deci	d	10	deca	da
10^{-2}	centi	c	10^{2}	hecto	h
10^{-3}	milli	m	10^{3}	kilo	k
10^{-6}	micro	μ	10^{6}	mega	M
10^{-9}	nano	n	10^{9}	giga	G
10^{-12}	pico	p	10^{12}	tera	T
10^{-15}	femto	f	10^{15}	peta	P
10^{-18}	atto	a	10^{18}	exa	E

ELECTRICITY AND MAGNETISM

In the following, σ = conductivity, $f = \omega/2\pi$ = radiation frequency, $\kappa_m = \mu/\mu_o$ and $\kappa_e = \varepsilon/\varepsilon_o$. Where subscripts are used, 1 denotes a conducting medium and 2 a propagating (lossless dielectric) medium. All units are mks unless otherwise specified.

Permittivity of free space	$\varepsilon_o = 8.8542 \times 10^{-12}$ farad/m.
Permeability of free space	$\mu_o = 4\pi \times 10^{-7} = 1.2566 \times 10^{-6}$ henry/m.
Resistance of free space	$R_o = (\mu_o/\varepsilon_o)^{1/2} = 376.73$ ohm
Relaxation time in a lossy medium	$\tau = \varepsilon/\sigma$ sec
Skin depth in a lossy medium	$\delta = (2/\omega\mu\sigma)^{1/2} = (\pi f\mu\sigma)^{-1/2}$ m
Wave impedance in lossy medium	$Z = [\mu/(\varepsilon + i\sigma/\omega)]^{1/2}$
Reflection coefficient at conducting surface (good only for $R \approx 1$)[7]	$R = 1 - 4.22 \times 10^{-4}(f\kappa_{m1}\kappa_{e2}/\sigma)^{1/2}$
Field at distance r from straight wire carrying current I	$B_\theta = \mu I/2\pi r$ tesla $= 0.2\ I/r$ gauss (r in cm)
Electromagnetic energy in volume V	$W = \frac{1}{2}\int_V dV(\underline{H}\cdot\underline{B} + \underline{E}\cdot\underline{D})$
Poynting's theorem	$\frac{\partial W}{\partial t} + \int_S \underline{N}\cdot d\underline{S} = \int_V dV\,\underline{J}\cdot\underline{E}$,

where S is the closed surface bounding V and the Poynting vector (energy flux across S) is $\underline{N} = \underline{E} \times \underline{H}$.

MAXWELL'S EQUATIONS

Name	Rationalized mks	Gaussian
Faraday's law	$\nabla \times E = -\frac{\partial B}{\partial t}$	$\nabla \times E = -\frac{1}{c}\frac{\partial B}{\partial t}$
Ampere's law	$\nabla \times H = \frac{\partial D}{\partial t} + J$	$\nabla \times H = \frac{1}{c}\frac{\partial D}{\partial t} + \frac{4\pi}{c} J$
Poisson's equation	$\nabla \cdot D = \rho$	$\nabla \cdot D = 4\pi\rho$
[Absence of magnetic monopoles]	$\nabla \cdot B = 0$	$\nabla \cdot B = 0$
Lorentz force on charge q	$q\left[E + v \times B\right]$	$q\left[E + \frac{1}{c} v \times B\right]$
Constitutive relations	$D = \epsilon E$ $B = \mu H$	$D = \epsilon E$ $B = \mu H$

In a plasma, $\mu \approx \mu_o$ (Gaussian units: $\mu \approx 1$). The permittivity satisfies $\epsilon \approx \epsilon_o$ (Gaussian: $\epsilon \approx 1$) provided all charge is regarded as free. Using the drift approximation $v_\perp = E \times B/B^2$ to calculate polarization charge density given rise to a dielectric constant $K = \epsilon/\epsilon_o = 1 + 36\pi \times 10^9\ \rho/B^2$ (mks) $= 1 + 4\pi\rho c^2/B^2$ (Gaussian), where ρ is the mass density.

ELECTROMAGNETIC FREQUENCY/WAVELENGTH BANDS

Nomenclature

E = Extreme

F = Frequency

H = High

L = Low

M = Medium

S = Super

U = Ultra

V = Very

Designation[e]	Frequency Range		Wavelength Range	
	Lower	Upper	Lower	Upper
ULF*		10 Hz	3 Mm	
ELF*	10 Hz	3 kHz	100 km	3 Mm
VLF	3 kHz	30 kHz	10 km	100 km
LF	30 kHz	300 kHz	1 km	10 km
MF	300 kHz	3 MHz	100 m	1 km
HF	3 MHz	30 MHz	10 m	100 m
VHF	30 MHz	300 MHz	1 m	10 m
UHF	300 MHz	3 GHz	10 cm	1 m
SHF‡	3 GHz	30 GHz	1 cm	10 cm
EHF	30 GHz	300 GHz	1 mm	1 cm
(Submillimeter)	300 GHz	3 THz	100 μ	1 mm
(Infrared)	3 THz			100 μ

*The boundary between ULF and ELF is variously defined.

‡The SHF (microwave) band is further subdivided approximately as follows:[9]

Band Designation	Frequency Range (GHz)		Wavelength Range (cm)	
	Lower	Upper	Lower	Upper
S	2.6	3.95	7.6	11.5
G	3.95	5.85	5.1	7.6
J	5.3	8.2	3.7	5.7
H	7.05	10.0	3.0	4.25
X	8.2	12.4	2.4	3.7
M	10.0	15.0	2.0	3.0
P	12.4	18.0	1.67	2.4
K	18.0	26.5	1.1	1.67
R	26.5	40.0	.75	1.1

AC CIRCUITS

For a resistance R, inductance L, and capacitance C in series with a voltage source $V = V_o \exp(j\omega t)$ (where $j = \sqrt{-1}$), the current I satisfies $I = dq/dt$, where

$$L \frac{d^2q}{dt^2} + R \frac{dq}{dt} + q/C = V \ .$$

Solutions are $q(t) = q_s + q_t$, where the steady state is $I_s = j\omega q_s = V/Z$ in terms of the impedance $Z = R + j(\omega L - 1/\omega C)$. For initial conditions $q(0) = q_o = \bar{q}_o + q_s(0)$, $I(0) = I_o$, the transients can be of three types, depending on $\Delta = R^2 - 4L/C$:

(a) Overdamped, $\Delta > 0$

$$q_t = \left[(I_o - \gamma_- \bar{q}_o) e^{\gamma_+ t} - (I_o - \gamma_+ \bar{q}_o) e^{\gamma_- t} \right] \Big/ (\gamma_+ - \gamma_-) ,$$

where

$$\gamma_\pm = \left(- R \pm \Delta^{\frac{1}{2}} \right) /2L \ ;$$

(b) Critically damped, $\Delta = 0$

$$q_t = \left[\bar{q}_o + (I_o + \gamma_R \bar{q}_o) t \right] e^{-\gamma_R t} \quad ,$$

where

$$\gamma_R = R/2L \ ;$$

(c) Underdamped, $\Delta < 0$

$$q_t = \left[\omega_1^{-1}(\gamma_R \bar{q}_o + I_o) \sin \omega_1 t + \bar{q}_o \cos \omega_1 t \right] e^{-\gamma_R t} ,$$

where

$$\omega_1 = \omega_o \left(1 - R^2 C/4L \right)^{1/2} , \quad \omega_o = (LC)^{-1/2} .$$

The quality of the circuit is $Q = \omega_o L/R$; ω_o is the resonant frequency. At $\omega = \omega_o$, $Z = R$. Instability results when L, R, C are not all of the same sign.

DIMENSIONLESS NUMBERS OF FLUID MECHANICS

Hundreds of dimensionless groups have been defined, particularly in chemical engineering and fluid dynamics.[10] The present collection includes only a small subset of the latter category. Notation is as follows:

c	speed of sound
c_p	specific heat at constant pressure (units cm^2/sec^2 - deg)
g	gravitational acceleration
H	vertical length scale
$k = \varkappa \rho c_p$	thermal conductivity (units g-cm/sec^3-deg)
L	horizontal* length scale
n	frequency of solid-body oscillation
r	radius of curvature of a pipe or channel
R	radius of pipe
α	Newton's law coefficient of heat transfer (units g/sec^3-deg)
β	volumetric expansion coefficient, $dV/V = \beta dT$
ΔT	applied temperature difference
$\varkappa$	thermal diffusivity (units cm^2/sec)
λ	collisional mean free path
$\mu = \rho \nu$	bulk viscosity
ν	kinematic viscosity
Ω	rotational velocity
ρ	mass density

Name	Symbol	Definition	Significance
Burgers	B	$(\beta g \Delta T)^{\frac{1}{2}}/2\Omega H^{\frac{1}{2}}$	effect of stratification/effect of rotation
Dean	D	$(UR/\nu)(R/r)^{\frac{1}{2}}$	transverse circulation due to curvature/longitudinal flow speed
Eckert	E	$U^2/c_p\Delta T$	adiabatic temperature increase/applied temperature difference
Ekman	Ek	$\nu/\Omega H^2$	viscous forces/coriolis forces
Froude	Fr	$\Omega^2 L/g$	centrifugal forces/gravitational forces
Grashof	G	$g\beta\Delta T L^3/\nu^2$	effects of bouyancy/viscous effects
Knudsen	Kn	λ/L	hydrodynamic time scale/collisional time scale
Mach	M	U/c	directed motion/thermal motion
Nusselt	N	$-\alpha L/k$	total thermal transfer/thermal conduction
Péclet	Pe	UL/κ	thermal diffusion time scale/free streaming time scale
Prandtl	Pr	ν/κ	momentum diffusion rate/heat diffusion rate
Rayleigh	Ra	$\beta g\Delta T H^3/\nu\kappa$	bouyancy forces/diffusive forces
Reynolds	Re	UL/ν	inertial forces/viscous forces
Richardson	Ri	$\beta g\Delta T H/U^2$	effects of bouyancy/effects of vertical shear
Rossby	Ro	$U/\Omega L$	inertial forces/coriolis forces
Stanton	S_x	$\alpha/\rho c_p U$	thermal convection/compressional heating
Strouhal	S	nL/U	oscillatory motion/directed motion

SHOCKS

Ideal gas jump conditions for density ρ, velocity u, pressure p, temperature T:

$$\frac{\rho_2}{\rho_1} = \frac{u_1}{u_2} = \frac{(\gamma + 1) M_1^2}{(\gamma - 1) M_1^2 + 2} \; ; \qquad \frac{p_2}{p_1} = \frac{2 \gamma M_1^2 - \gamma + 1}{\gamma + 1} \; ;$$

$$\frac{T_2}{T_1} = \frac{\left[2 \gamma M_1^2 - \gamma + 1\right]\left[(\gamma - 1) M_1^2 + 2\right]}{(\gamma + 1)^2 M_1^2} \; ,$$

where 1, 2 denote quantities in front of and behind the shock respectively in the shock frame; $M_1 = u_1/c_{s1}$ = *Mach number in region* 1, $c_s^2 = \gamma kT/\rho$; γ = ratio of specific heats = $(f + 2)/f$, where f is the number of degrees of freedom. Entropy change across the shock is

$$\Delta S = S_2 - S_1 = c_v \, \ell n \left[\frac{p_2}{p_1}\left(\frac{\rho_1}{\rho_2}\right)^{\gamma}\right] \; ,$$

where c_v is the specific heat at constant volume. For weak shocks,

$$\Delta S \rightarrow c_v \frac{2 \gamma(\gamma - 1)}{3 (\gamma + 1)^2} (M_1^2 - 1)^3 \; .$$

Evidently in the limit $M_1 \rightarrow \infty$, $p_2/\rho_1 \rightarrow (\gamma + 1)/(\gamma - 1)$ ($= 4$ for $\gamma = 5/3$), while the temperature and pressure jumps become infinite.

PLASMA DISPERSION FUNCTION [11]

Definition

$$Z(\zeta) = \pi^{-\frac{1}{2}} \int_{-\infty}^{\infty} dt \; \exp(-t^2)/(t - \zeta) \qquad (\mathrm{Im}\zeta > 0)$$

$$= 2\, i \exp(-\zeta^2) \int_{-\infty}^{i\zeta} dt \; \exp(-t^2) \; .$$

Physically $\zeta = x + iy$ is the ratio of phase to thermal velocity.

Differential equation

$$\frac{dZ}{d\zeta} = -2\left[1 + \zeta Z\right], \quad Z(0) = i\pi^{\frac{1}{2}} \; ;$$

$$\frac{d^2 Z}{d\zeta^2} + 2\zeta \frac{dZ}{d\zeta} + 2Z = 0 \; .$$

Real argument ($y = 0$)

$$Z(x) = i\pi^{\frac{1}{2}} \exp(-x^2) - 2x\, Y(x) \quad ,$$

$$Y(x) = x^{-1} \exp(-x^2) \int_0^x dt \; \exp(t^2) .$$

Imaginary argument

$$Z(iy) = i\pi^{\frac{1}{2}} \exp(y^2) \left[1 - \mathrm{erf}(y)\right] = i\pi^{\frac{1}{2}} \exp(y^2) \mathrm{erfc}(y)$$

Power series (small argument)

$$Z(\zeta) = i\pi^{\frac{1}{2}} \exp(-\zeta^2) - 2\zeta\left[1 - 2\zeta^2/3 + 4\zeta^4/15 - 8\zeta^6/105 + \cdots\right].$$

Asymptotic series (large argument, $x > 0$) [12]

$$Z(\zeta) = i\pi^{\frac{1}{2}}\sigma \exp(-\zeta^2) - \zeta^{-1}\left[1 + 1/2\,\zeta^{-2} + 3/4\,\zeta^{-4} + 15/8\,\zeta^{-6} + \ldots\right]$$

$$\sigma = \begin{cases} 0 & y > 1/x \\ 1 & |y| < 1/x \\ 2 & y < -1/x \end{cases}$$

Symmetry properties ($\zeta^* = x - iy$)

$$Z(\zeta^*) = -\left[Z(-\zeta)\right]^*$$

For $y > 0$, $Z(x - iy) = \left[Z(x + iy)\right]^* + 2\,i\pi^{\frac{1}{2}} \exp\left[-(x - iy)^2\right].$

Two-pole approximations [13]

For ζ in upper half plane (good except when $y < \pi^{\frac{1}{2}} x^2 e^{-x^2}$, $x \gg 1$)

$$Z(\zeta) \approx \frac{0.50 + 0.81\,i}{a - \zeta} - \frac{0.50 - 0.81\,i}{a^* - \zeta} \quad ;$$

$$Z'(\zeta) \approx \frac{0.50 + 0.86\,i}{(b - \zeta)^2} + \frac{0.50 - 0.86\,i}{(b^* - \zeta)^2} \quad ;$$

$$a = 0.51 - 0.81\,i \quad ,$$
$$b = 0.48 - 0.91\,i \quad .$$

FUNDAMENTAL PLASMA PARAMETERS

All quantities are in Gaussian units except temperature (T_e, T_i, T) expressed in eV and ion mass (m_i) expressed in units of proton mass, $\mu = m_i/m_p$; Z is charge state; k is Boltzmann's constant; K is wavenumber.

<u>Frequencies</u>

electron gyrofrequency $f_{ce} = \omega_{ce}/2\pi = 2.80 \times 10^6\ B$ Hz

$\omega_{ce} = eB/m_e c = 1.76 \times 10^7\ B$ rad/sec

ion gyrofrequency $f_{ci} = \omega_{ci}/2\pi = 1.52 \times 10^3\ Z\mu^{-1}\ B$ Hz

$\omega_{ci} = eB/m_i c = 9.58 \times 10^3\ Z\mu^{-1}\ B$ rad/sec

hybrid gyrofrequency $f_{LH} = \omega_{LH}/2\pi = 6.53 \times 10^4\ Z^{\frac{1}{2}}\ \mu^{-\frac{1}{2}}\ B$ Hz

$\omega_{LH} = (\omega_{ce}\omega_{ci})^{\frac{1}{2}}$

$= 4.10 \times 10^5\ Z^{\frac{1}{2}}\ \mu^{-\frac{1}{2}}\ B$ rad/sec

electron plasma frequency $f_{pe} = \omega_{pe}/2\pi = 8.98 \times 10^3\ n_e^{\frac{1}{2}}$ Hz

$\omega_{pe} = (4\pi n_e e^2/m_e)^{\frac{1}{2}}$

$= 5.64 \times 10^4\ n_e^{\frac{1}{2}}$ rad/sec

ion plasma frequency $f_{pi} = \omega_{pi}/2\pi = 2.10 \times 10^2\ Z\mu^{-\frac{1}{2}}\ n_i^{\frac{1}{2}}$ Hz

$\omega_{pi} = (4\pi n_i Z^2 e^2/m_i)^{\frac{1}{2}}$

$= 1.32 \times 10^3\ Z\mu^{-\frac{1}{2}}\ n_i^{\frac{1}{2}}$ rad/sec

electron trapping rate $\quad \nu_{Te} = \left(\frac{eKE}{m_e}\right)^{\frac{1}{2}} = 7.26 \times 10^8\ K^{\frac{1}{2}} E^{\frac{1}{2}}\ \sec^{-1}$

ion trapping rate $\quad \nu_{Ti} = \left(\frac{eKE}{m_i}\right)^{\frac{1}{2}} = 1.69 \times 10^7\ K^{\frac{1}{2}} E^{\frac{1}{2}} \mu^{-\frac{1}{2}}\ \sec^{-1}$

Velocities

electron thermal velocity $\quad v_{Te} = (kT_e/m_e)^{\frac{1}{2}} = 4.19 \times 10^7\ T_e^{\frac{1}{2}}$ cm/sec

ion thermal velocity $\quad v_{Ti} = (kT_i/m_i)^{\frac{1}{2}} = 9.79 \times 10^5\ \mu^{-\frac{1}{2}}\ T_i^{\frac{1}{2}}$ cm/sec

ion sound velocity $\quad c_s = (kT_e/m_i)^{\frac{1}{2}} = 9.79 \times 10^5\ \mu^{-\frac{1}{2}}\ T_e^{\frac{1}{2}}$ cm/sec

Alfvén velocity $\quad v_A = B/(4\pi n_i m_i)^{\frac{1}{2}}$

$$= 2.18 \times 10^{11}\ \mu^{-\frac{1}{2}}\ n_i^{-\frac{1}{2}}\ B \quad \text{cm/sec}$$

With this definition of thermal velocity, the Debye length and plasma frequency for a species j are connected by

$$\omega_{pj}\lambda_{Dj} = v_{Tj}\ ,$$

where λ_{Dj} is obtained by substituting T_j for T in the definition below. Some individuals, however, prefer the definition

$$v'_{Tj} = (2\ kT_j/m_j)^{\frac{1}{2}}\ ,$$

according to which the electron and ion thermal velocities and ion sound velocity become

$$v'_{Te} = (2\ kT_e/m_e)^{\frac{1}{2}} = 5.93 \times 10^7\ T_e^{\frac{1}{2}}\ \text{cm/sec};$$

$$v'_{Ti} = (2\ kT_i/m_i)^{\frac{1}{2}} = 1.38 \times 10^6\ (T_i/\mu)^{\frac{1}{2}}\ \text{cm/sec};$$

$$c'_s = (2\ kT_e/m_i)^{\frac{1}{2}} = 1.38 \times 10^6\ (T_e/\mu)^{\frac{1}{2}}\ \text{cm/sec}.$$

The Debye length and gyroradii may or may not be redefined accordingly.

Lengths

electron deBroglie length	$\lambda\!\!\!^{-} = \hbar/(m_e kT_e)^{\frac{1}{2}} = 2.76 \times 10^{-8}\ T_e^{-\frac{1}{2}}$ cm
classical distance of minimum approach	$e^2/kT = 1.44 \times 10^{-7}\ T^{-1}$ cm
electron gyroradius	$r_e = v_{Te}/\omega_{ce} = 2.38\ T_e^{\frac{1}{2}}\ B^{-1}$ cm
ion gyroradius	$r_i = v_{Ti}/\omega_{ci} = 1.02 \times 10^2\ \mu^{\frac{1}{2}}\ Z^{-\frac{1}{2}}\ T_i^{\frac{1}{2}}\ B^{-1}$ cm
plasma skin depth	$c/\omega_{pe} = 5.31 \times 10^5\ n^{-\frac{1}{2}}$ cm
Debye length	$\lambda_D = (kT/4\pi ne^2)^{\frac{1}{2}} = 7.43 \times 10^2\ T^{\frac{1}{2}}\ n^{-\frac{1}{2}}$ cm

Dimensionless

(electron/proton mass ratio)$^{\frac{1}{2}}$	$(m_e/m_p)^{\frac{1}{2}} = 2.33 \times 10^{-2} = 1/42.9$
number of particles in Debye sphere	$\frac{4\pi}{3} n\lambda_D^3 = 1.72 \times 10^9\ T^{3/2}\ n^{-\frac{1}{2}}$

Alfvén velocity/speed of light	$v_A/c = 7.28\ \mu^{-\frac{1}{2}}\ n_i^{-\frac{1}{2}}\ B$
magnetic/ion rest energy ratio	$B^2/8\pi n_i m_i c^2 = 26.5\ \mu^{-1}\ n_i^{-1}\ B^2$
electron plasma/gyrofrequency ratio	$\omega_{pe}/\omega_{ce} = 3.21 \times 10^{-3}\ n_e^{\frac{1}{2}}\ B^{-1}$
ion plasma/gyrofrequency ratio	$\omega_{pi}/\omega_{ci} = 0.137\ \mu^{\frac{1}{2}}\ n_i^{\frac{1}{2}}\ B^{-1}$
thermal/magnetic energy ratio	$\beta = 8\pi nT/B^2 = 4.03 \times 10^{-11}\ nT\ B^{-2}$

Miscellaneous

Bohm diffusion coefficient $\quad D_B = \dfrac{ckT}{16\ eB} = 6.25 \times 10$[illegible] $TB^{-1}\ cm^2\ sec^{-1}$

Spitzer resistivity (transverse)

$$\eta_\perp = 1.15 \times 10^{-14}\ Z\ \ell n\Lambda\ T^{-3/2}\ sec$$

$$= 1.03 \times 10^{-2}\ Z\ \ell n\Lambda\ T^{-3/2}\ ohm\text{-}cm$$

The anomalous collision rate due to low frequency ion sound turbulence is [14]

$$\nu^* \approx \omega_{pe} W/kT = 5.64 \times 10^4\ n^{-1/2}\ W/kT\ sec^{-1},$$

where W is the total energy of waves with $\omega/K < v_{Ti}$.

Magnetic pressure is given by

$$P = 3.98 \times 10^{-2}\ B^2\ \ dyne/cm^2,$$

when P and B are expressed in cgs units. Thus the pressure equivalent of 10 kG is

$$P(10\ kG) = 3.98 \times 10^{6}\ dyne/cm^2 = 3.93\ atm.$$

APPROXIMATE MAGNITUDES IN SOME TYPICAL PLASMAS

Plasma Type	n cm^{-3}	T eV	ω_{pe} sec^{-1}	λ_D cm	$n\lambda_D^3$	ν_{ei} sec^{-1}
Interstellar gas	1	1	6×10^4	7×10^2	4×10^8	7×10^{-5}
Gaseous nebula	10^3	1	2×10^6	20	10^7	6×10^{-2}
Solar corona	10^6	10^2	6×10^7	7	4×10^8	6×10^{-2}
Diffuse hot plasma	10^{12}	10^2	6×10^{10}	7×10^{-3}	4×10^5	40
Solar atmosphere, gas discharge	10^{14}	1	6×10^{11}	7×10^{-5}	40	2×10^9
Warm plasma	10^{14}	10	6×10^{11}	2×10^{-4}	10^3	10^7
Hot plasma	10^{14}	10^2	6×10^{11}	7×10^{-4}	4×10^4	4×10^6
Thermonuclear plasma	10^{15}	10^4	2×10^{12}	2×10^{-3}	10^7	5×10^4
Theta pinch	10^{16}	10^2	6×10^{12}	7×10^{-5}	4×10^3	3×10^8
Dense hot plasma	10^{18}	10^2	6×10^{13}	7×10^{-6}	4×10^2	2×10^{10}
Laser plasma	10^{20}	10^2	6×10^{14}	7×10^{-7}	40	2×10^{12}

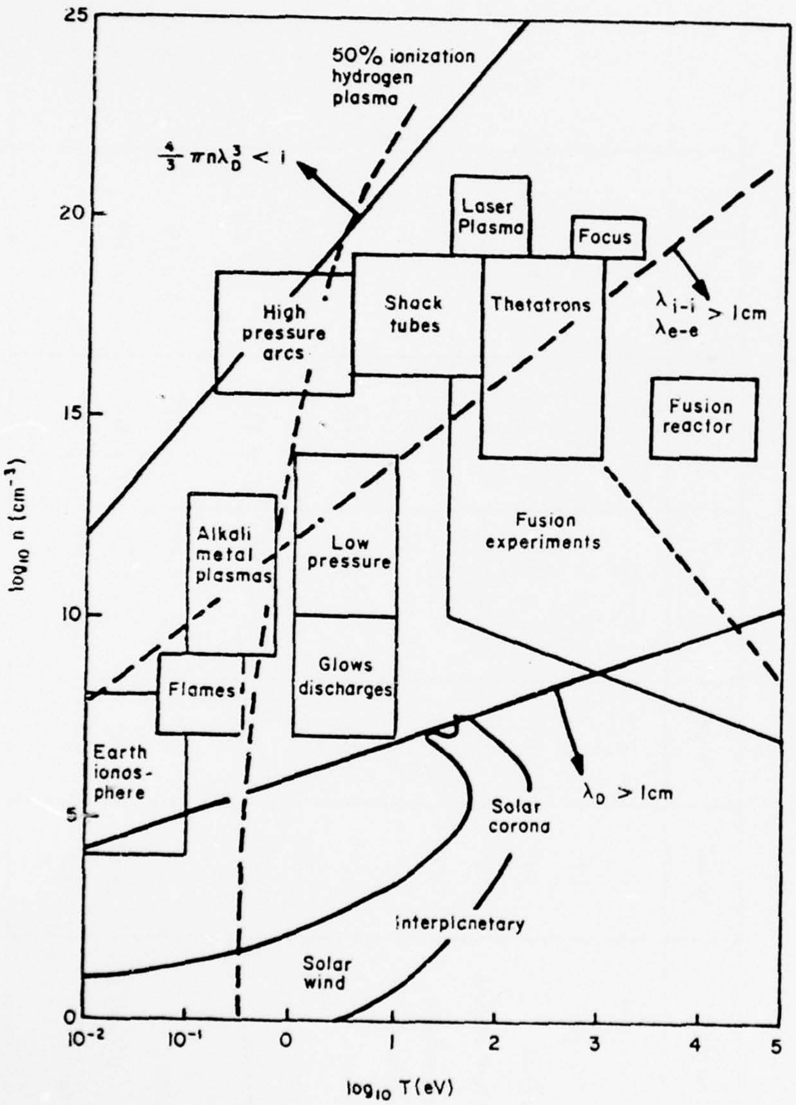

Fig. 1 — Information of a similar nature is depicted graphically[15]

COLLISIONS AND TRANSPORT

Temperatures are in eV; the corresponding value of Boltzmann's constant is $k = 1.60 \times 10^{-12}$ erg/eV; masses μ, μ' are in units of the proton mass; $e_\alpha = Z_\alpha e$ is the charge of species α. All other units are cgs except where noted.

Relaxation rates [18]

Rates are associated with four relaxation processes arising from the interaction of test particles (labeled α) streaming through a background of field particles (labeled β):

slowing down $$\frac{d\mathbf{v}_\alpha}{dt} = -\nu_s^{\alpha/\beta}\mathbf{v}_\alpha \quad ;$$

transverse diffusion $$\frac{d}{dt}(\mathbf{v}_\alpha - \bar{\mathbf{v}}_\alpha)_\perp^2 = \nu_\perp^{\alpha/\beta} v_\alpha^2 \quad ;$$

parallel diffusion $$\frac{d}{dt}(\mathbf{v}_\alpha - \bar{\mathbf{v}}_\alpha)_\parallel^2 = \nu_\parallel^{\alpha/\beta} v_\alpha^2 \quad ;$$

energy loss $$\frac{d}{dt} v_\alpha^2 = -\nu_\epsilon^{\alpha/\beta} v_\alpha^2 \quad ,$$

where the averages are performed over an ensemble of test particles and a Maxwellian field particle distribution.

The exact formulas may be written

$$\nu_s^{\alpha/\beta} = (1 + m_\alpha/m_\beta)\psi(x^{\alpha/\beta})\ \nu_o^{\alpha/\beta}\ \sec^{-1} ;$$

$$\nu_\perp^{\alpha/\beta} = 2\left[\psi(x^{\alpha/\beta})(1 - 1/2x^{\alpha/\beta}) + \psi'(x^{\alpha/\beta})\right]\nu_o^{\alpha/\beta}\ \sec^{-1} ;$$

$$\nu_\parallel^{\alpha/\beta} = \left[\psi(x^{\alpha/\beta})/x^{\alpha/\beta}\right]\nu_o^{\alpha/\beta}\ \sec^{-1} ;$$

$$\nu_\epsilon^{\alpha/\beta} = 2\left[(m_\alpha/m_\beta)\psi(x^{\alpha/\beta}) - \psi'(x^{\alpha/\beta})\right]\nu_o^{\alpha/\beta}\ \sec^{-1} ,$$

where

$$\nu_o^{\alpha/\beta} = 4\pi e_\alpha^2 e_\beta^2 \lambda_{\alpha\beta} n_\beta / m_\alpha^2 v_\alpha^3\ \sec^{-1} ;$$

$$x^{\alpha/\beta} = \frac{1}{2}\, m_\beta v_\alpha^2 / kT_\beta ;$$

$$\psi(x) = \frac{2}{\sqrt{\pi}} \int_0^x dt \sqrt{t}\, e^{-t} ;$$

$$\psi'(x) = d\psi/dx ,$$

and $\lambda_{\alpha\beta} = \ln \Lambda_{\alpha\beta}$ is the Coulomb logarithm (see below). The limiting forms of ν_s, $\nu_\perp$ and $\nu_\parallel$ are given in the following table. The two expressions shown for each rate hold for very slow ($x^{\alpha/\beta} \ll 1$) and very fast ($x^{\alpha/\beta} \gg 1$) test particles, respectively. Test particle energy ϵ and field particle temperature T are both in eV; $\mu = m_i/m_p$, where m_p is the proton mass; for electron-electron and ion-ion encounters, field particle quantities are distinguished by a prime.

Electron-electron

$$\nu_s^{e/e'}/n_{e'}\lambda_{ee'} \approx 5.8 \times 10^{-6}\ T^{-3/2} \rightarrow 7.7 \times 10^{-6}\ \varepsilon^{-3/2}$$

$$\nu_\perp^{e/e'}/n_{e'}\lambda_{ee'} \approx 5.8 \times 10^{-6}\ T^{-1/2}\ \varepsilon^{-1} \rightarrow 7.7 \times 10^{-6}\ \varepsilon^{-3/2}$$

$$\nu_\parallel^{e/e'}/n_{e'}\lambda_{ee'} \approx 2.9 \times 10^{-6}\ T^{-1/2}\ \varepsilon^{-1} \rightarrow 3.9 \times 10^{-6}\ T\varepsilon^{-5/2}$$

Electron-ion

$$\nu_s^{e/i}/n_i Z^2 \lambda_{ei} \approx 0.23\ \mu^{3/2} T^{-3/2} \rightarrow 3.9 \times 10^{-6}\ \varepsilon^{-3/2}$$

$$\nu_\perp^{e/i}/n_i Z^2 \lambda_{ie} \approx 2.5 \times 10^{-4}\ \mu^{1/2} T^{-1/2} \varepsilon^{-1} \rightarrow 7.7 \times 10^{-6}\ \varepsilon^{-3/2}$$

$$\nu_\parallel^{e/i}/n_i Z^2 \lambda_{ie} \approx 1.2 \times 10^{-4}\ \mu^{1/2} T^{-1/2} \varepsilon^{-1} \rightarrow 2.1 \times 10^{-9}\ \mu^{-1} T\varepsilon^{-5/2}$$

Ion-electron

$$\nu_s^{i/e}/n_e \lambda_{ie} \approx 1.6 \times 10^{-9}\ \mu^{-1} T^{-3/2} \rightarrow 1.7 \times 10^{-4}\ \mu^{1/2}\ \varepsilon^{-3/2}$$

$$\nu_\perp^{i/e}/n_e \lambda_{ie} \approx 3.2 \times 10^{-9}\ \mu^{-1}\ T^{-1/2}\ \varepsilon^{-1} \rightarrow 1.8 \times 10^{-7}\ \mu^{-1/2}\ \varepsilon^{-3/2}$$

$$\nu_\parallel^{i/e}/n_e \lambda_{ie} \approx 1.6 \times 10^{-9}\ \mu^{-1}\ T^{-1/2} \varepsilon^{-1} \rightarrow 1.7 \times 10^{-4}\ \mu^{1/2} T\varepsilon^{-5/2}$$

Ion-ion

$$\nu_s^{i/i'}/n_{i'}Z^2Z'^2\lambda_{ii'} \approx 6.8 \times 10^{-8} \frac{\mu'^{1/2}}{\mu} \left(1 + \frac{\mu'}{\mu}\right) T^{-3/2}$$

$$\rightarrow 9.0 \times 10^{-8} \mu^{-1/2} \left(1 + \frac{\mu}{\mu'}\right) \varepsilon^{-3/2}$$

$$\nu_\perp^{i/i'}/n_{i'}Z^2Z'^2\lambda_{ii'} \approx 1.4 \times 10^{-7} \mu'^{1/2}\mu^{-1}T^{-1/2}\varepsilon^{-1}$$

$$\rightarrow 1.8 \times 10^{-7} \mu^{-1/2}\varepsilon^{-3/2}$$

$$\nu_\parallel^{i/i'}/n_{i'}Z^2Z'^2\lambda_{ii'} \approx 6.8 \times 10^{-8} \mu'^{1/2}\mu^{-1}T^{-1/2}\varepsilon^{-1}$$

$$\rightarrow 9.0 \times 10^{-8} \mu^{1/2}\mu'^{-1}T\varepsilon^{-5/2}$$

In the same limits, the energy transfer rate follows from the identity

$$\nu_\varepsilon = 2\,\nu_s - \nu_\perp - \nu_\parallel ,$$

except for the case of fast electrons or fast ions scattered by ions, where the leading terms cancel. Here the appropriate forms are

$$\nu_\varepsilon^{e/i} \rightarrow 4.2 \times 10^{-9}\, n_i Z^2 \lambda_{ei} \Big[\varepsilon^{-3/2}\mu^{-1}$$

$$- 8.9 \times 10^4 (\mu/T)^{1/2}\varepsilon^{-1} \exp(-\,1836\,\mu\varepsilon/T)\Big] \ \sec^{-1}$$

and

$$\nu_\varepsilon^{i/i'} \rightarrow 1.8 \times 10^{-7}\, n_i Z^2 Z'^2 \lambda_{ii'} \Big[\varepsilon^{-3/2}\mu^{1/2}/\mu'$$

$$- 1.1\,(\mu'/T)^{1/2}\varepsilon^{-1}\exp(-\,\mu'\varepsilon/T)\Big]\ \sec^{-1}\ .$$

In general, the energy transfer rate $\nu_\varepsilon^{\alpha/\beta}$ is positive for $\varepsilon > \varepsilon_\alpha^*$ and negative for $\varepsilon < \varepsilon_\alpha^*$, where $x^* = (m_\beta/m_\alpha)\ \varepsilon_\alpha^*/T_\beta$ is the solution of $m_\beta/m_\alpha = \psi(x^*)\psi'(x^*)$.

The ratio $\varepsilon_\alpha^*/T_\beta$ is given for a number of specific α, β in the following table:

α/β	i/e	e/e	i/i	e/p	e/D	e/T, e/He^3	e/He^4
$\frac{\varepsilon_\alpha^*}{T_\beta}$	1.5	.98	.98	4.8×10^{-3}	2.6×10^{-3}	1.8×10^{-3}	1.4×10^{-3}

The collision rates for various types of encounter are represented graphically in Fig. 2.[17]

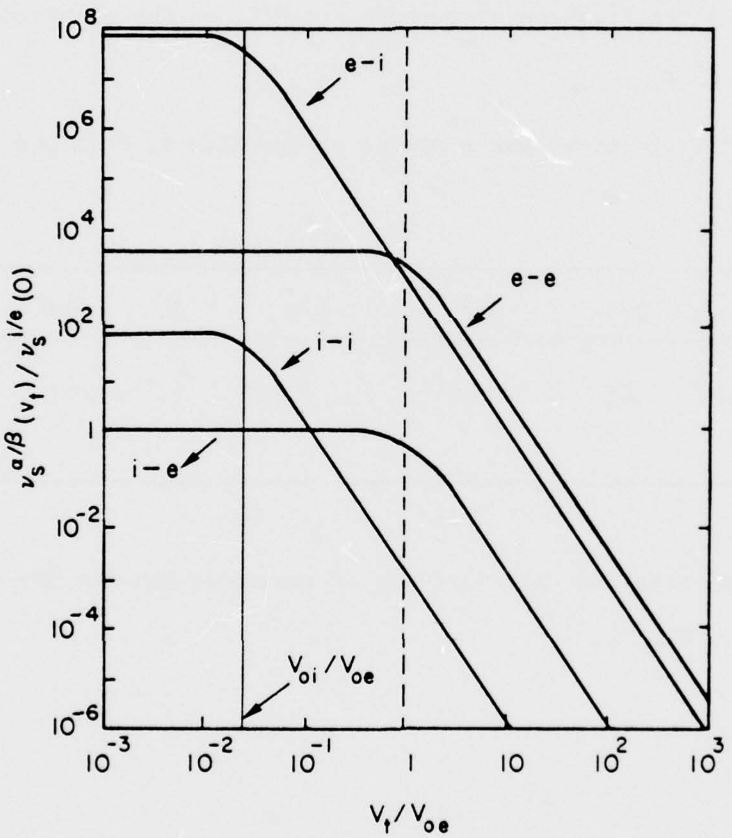

Fig. 2 - Test particle showing-down rates $\nu_s^{\alpha/\beta}$, scaled by the ion-electron rate $\nu_s^{i/e}(0)$ in the limit of zero relative velocity v_t, as a function of v_t divided by the electron thermal velocity.

Thermal equilibration

If the components of a plasma have different temperatures, but no relative drift, equilibration is described by

$$\frac{dT_\alpha}{dt} = \sum_\beta \bar{\nu}_\epsilon^{\alpha/\beta} (T_\beta - T_\alpha) ,$$

where

$$\bar{\nu}_\epsilon^{\alpha/\beta} = 1.8 \times 10^{-19} \frac{(m_\alpha m_\beta)^{1/2} Z_\alpha^2 Z_\beta^2 n_\beta \lambda_{\alpha\beta}}{(m_\alpha T_\beta + m_\beta T_\alpha)^{3/2}} \quad sec^{-1} .$$

For electrons and ions with $T_e \sim T_i = T$, this implies

$$\frac{\bar{\nu}_\epsilon^{e/i}}{n_i} = \frac{\bar{\nu}_\epsilon^{i/e}}{n_e} = 3.2 \times 10^{-9} \frac{Z^2 \lambda}{\mu T^{3/2}} \quad cm^3/sec.$$

Temperature anisotropy

Isotropization is described by

$$\frac{dT_\perp}{dt} = -\left(\frac{1}{2}\right)\frac{dT_\parallel}{dt} = -\nu_T^\alpha (T_\perp - T_\parallel) ,$$

where, if $A = T_\perp / T_\parallel - 1 > 0$,

$$\nu_T^\alpha = \frac{2\sqrt{\pi}\, e_\alpha^4 n_\alpha \lambda}{m_\alpha^{1/2} (kT_\parallel)^{3/2}} A^{-2} \left[-3 + (A + 3) \frac{\tan^{-1} A^{1/2}}{A^{1/2}} \right] \quad sec^{-1} ,$$

if $A < 0$, $\tan^{-1} A^{1/2}/A^{1/2}$ is replaced by $\tanh^{-1}(-A)^{1/2}/(-A)^{1/2}$.

For $T_\perp \approx T_\parallel = T$,

$$\nu_T^e = 8.2 \times 10^{-7}\, n\lambda T^{-3/2} \qquad \sec^{-1} \quad ;$$

$$\nu_T^i = 1.9 \times 10^{-8}\, n\lambda Z^2/\mu T^{3/2} \qquad \sec^{-1} \quad .$$

Coulomb logarithm

For test particles of mass m_α, charge $e_\alpha = Z_\alpha e$, scattering off field particles of mass m_β, charge $e_\beta = Z_\beta e$, the Coulomb logarithm is defined as $\lambda \equiv \ln \Lambda = \ln(r_{max}/r_{min})$. Here r_{min} is the larger of $e_\alpha e_\beta/(m_{\alpha\beta}\bar{u}^2)$ and $\hbar/(2\, m_{\alpha\beta}\bar{u})$, averaged over both particle velocity distributions, where $m_{\alpha\beta} = m_\alpha m_\beta/(m_\alpha + m_\beta)$ and $\underline{u} = \underline{v}_\alpha - \underline{v}_\beta$; $r_{max} = (4\pi \Sigma\, n_\gamma e_\gamma^2/kT_\gamma)^{-1/2}$, where the summation extends over all species γ for which $\bar{u}^2 < v_{T\gamma}^2$, with $v_{T\gamma} = (kT_\gamma/m_\gamma)^{1/2}$. If this inequality cannot be satisfied or if either $\bar{u}\,\omega_{c\alpha}^{-1} < r_{max}$ or $\bar{u}\,\omega_{c\beta}^{-1} < r_{max}$, the theory breaks down. Typically $\lambda \approx 10 - 20$. Corrections to the transport coefficients are $O(\lambda^{-1})$, hence the theory is good only to $\sim 10\%$ and fails when $\lambda \sim 1$.

The following cases are of particular interest.

(a) Thermal electron-electron collisions

$$\lambda_{ee} = 23 - \ln(n_e^{1/2}\, T_e^{-3/2}) \qquad T_e \leq 10 \text{ eV}$$

$$= 24 - \ln(n_e^{1/2}\, T_e^{-1}) \qquad T_e \geq 10 \text{ eV}$$

(b) Electron-ion collisions

$$\lambda_{ei} = \lambda_{ie} = 23 - \ln\left(n_e^{1/2} Z T_e^{-3/2}\right), \qquad 10 \text{ eV} > T_e > T_i m_e/m_i ;$$

$$= 24 - \ln\left(n_e^{1/2} T_e^{-1}\right), \qquad T_e > 10 \text{ eV} > T_i m_e/m_i ;$$

$$= 30 - \ln\left(n_i^{1/2} T_i^{-3/2} Z^2 \mu^{-1}\right), \qquad T_i > T_e m_i/m_e .$$

(c) Mixed ion-ion collisions

$$\lambda_{ii'} = \lambda_{i'i} = 23 - \ln\left[\frac{ZZ'(\mu + \mu')}{\mu T_{i'} + \mu' T_i}\left(\frac{n_i Z^2}{T_i} + \frac{n_{i'} Z'^2}{T_{i'}}\right)^{1/2}\right]$$

(d) Counterstreaming ions (relative velocity $v_D = \beta_D c$) in the presence of warm electrons, $kT_e/m_e > v_D^2 > kT_i/m_i,\ kT_{i'}/m_{i'}$,

$$\lambda_{ii'} = \lambda_{i'i} = 35 - \ln\left[\frac{ZZ'(\mu + \mu')}{\mu\mu'\beta_D^2}\left(\frac{n_e}{T_e}\right)^{1/2}\right].$$

Examples

1. Find the slowing-down and energy-transfer rates for a 100 keV deuteron beam in a plasma consisting of 1 keV electrons and 100 eV tritons, each having number density 10^{14} cm^{-3}.

Answer: The Coulomb logarithms are

$$\lambda_{De} = 24 - \ln\left[(10^{14})^{1/2}(10^{3})^{-1}\right]$$

$$= 15$$

from formula (b), and

$$\lambda_{DT} = 35 - \ln\left[\frac{(5/6)}{(2.2 \times 10^{8}/3 \times 10^{10})^{2}}\left(\frac{10^{14}}{10^{2}}\right)^{1/2}\right]$$

$$= 12$$

from formula (d). The beam velocity $v_D = 2.2 \times 10^{8}$ cm/sec satisfies $\bar{v}_T \ll v_D \ll \bar{v}_e$, where $\bar{v}_T$ and $\bar{v}_e$ are the respective thermal velocities. It is therefore appropriate to use the first form for $\nu_s^{D/e}$ and the second for $\nu_s^{D/T}$. Hence

$$\nu_s^{D/e} = (1.6 \times 10^{-9})(2)^{-1}(10^{3})^{-3/2}(10^{14})(15)$$

$$= 37 \text{ sec}^{-1},$$

and

$$\nu_s^{D/T} = (9 \times 10^{-8})(2)^{-1/2}(1 + 2/3)(10^{5})^{-3/2}(10^{14})(12)$$

$$= 3.9 \text{ sec}^{-1}$$

The net rate of transfer of energy to the electrons is

$$\nu_\epsilon^{D/e} = 2\,\nu_s^{D/e} - \nu_\perp^{D/e} - \nu_\parallel^{D/e} = \nu_s^{D/e} = 37\ \sec^{-1}\ ,$$

while the corresponding rate for tritons follows from the second formula for fast ions:

$$\nu_\epsilon^{D/T} = (1.8 \times 10^{-7})(10^{14})(12)\left[(10^5)^{-3/2}\,(2)^{1/2}\,(3)^{-1}\right.$$

$$\left. - (1.1)(3/10^2)^{1/2}(10^5)^{-1}\ \exp(-\,3 \times 10^5/10^2)\right]$$

$$= 3.1\ \sec^{-1}.$$

Evidently, for the parameters given, electron collisions are about an order of magnitude more effective in both processes.

2. A hydrogen plasma has $T_e = 500$ eV, $T_i = 100$ eV and $n = 10^{10}\ cm^{-3}$. If it contains an admixture of 1% $Li^7 (Z = 3)$ ions with $T_{Li} = 25$ eV, is the net effect of collisions to heat or to cool the H^+ ions? What happens if $T_e = 5$ keV is used?

Answer: The Coulomb logarithms are

$$\lambda_{H^+e} = 24 - \ln\left[(10^{10})^{1/2}(500)^{-1/2}\right]$$

$$= 16,$$

and

$$\lambda_{H^+Li^{+++}} = 23 - \ln\left\{\frac{(3)(8)}{25 + (7)(100)}\left[\frac{10^{10}}{100}\right.\right.$$

$$\left.\left. + \frac{(10^8)(9)}{25}\right]\right\} = 8.0$$

from (b) and (c), respectively. The thermal equilibration rates are thus

$$\bar{\nu}_\varepsilon^{H^+/e} = 3.2 \times 10^{-9} \, n\lambda_{H^+e} \, T_e^{-3/2}$$

$$= 4.5 \times 10^{-2} \text{ sec}^{-1},$$

and

$$\bar{\nu}_\varepsilon^{H^+/Li^{+++}} = \frac{1.8 \times 10^{-19}}{(1.7 \times 10^{-24})^{1/2}} \; \frac{(7)^{1/2}(3^2)(10^{-2}n)\lambda_{H^+Li^{+++}}}{\left[25 + (7)(100)\right]^{3/2}}$$

$$= .12 \text{ sec}^{-1}.$$

Hence

$$\frac{dT_{H^+}}{db} = \bar{\nu}_\varepsilon^{H^+/Li^{+++}} \left[25 - 100\right] + \bar{\nu}_\varepsilon^{H^+/e} \left[500 - 100\right]$$

$$= -9 + 18 = 9 \text{ eV/sec} > 0.$$

Evidently as $T_e \to \infty$ the electron contribution to $\frac{dT_{H^+}}{dt}$ vanishes $\sim T_e^{-1/2}$ and for sufficiently large T_e, $\frac{dT_{H^+}}{dt} < 0$. Thus with $T_e = 5$ keV,

$$\bar{\nu}_\varepsilon^{H^+/e} = 1.4 \times 10^{-3} \text{ sec}^{-1}$$

and

$$\frac{dT_{H^+}}{dt} = (.12)\left[25 - 100\right] + (1.4 \times 10^{-3})\left[5000 - 100\right]$$

$$= -2 \text{ eV/sec.}$$

Hence heating the electrons cools the H^+ ions!

Fokker-Planck equation

$$\frac{Df^\alpha}{Dt} = \frac{\partial f^\alpha}{\partial t} + \underline{v}\cdot\underline{\nabla} f^\alpha + \underline{F}\cdot\nabla_{\underline{v}} f^\alpha = \left(\frac{\partial f^\alpha}{\partial t}\right)_{coll} ,$$

where $\underline{F}$ is an external force field. The general form of the collision integral is $\left(\frac{\partial f^\alpha}{\partial t}\right)_{coll} = -\sum_\beta \nabla_{\underline{v}}\cdot \underline{J}^{\alpha/\beta}$ with

$$\underline{J}^{\alpha/\beta} = 2\pi\lambda_{\alpha\beta}\frac{e_\alpha^2 e_\beta^2}{m_\alpha}\int d^3v'(u^2\underline{\underline{I}} - \underline{u}\,\underline{u})u^{-3}$$

$$\left\{\frac{1}{m_\beta} f^\alpha(\underline{v})\nabla_{\underline{v}'} f^\beta(\underline{v}') - \frac{1}{m_\alpha} f^\beta(\underline{v}')\nabla_{\underline{v}} f^\alpha(\underline{v})\right\}$$

(Landau form) where $\underline{u} = \underline{v}' - \underline{v}$ and $\underline{\underline{I}}$ is the unit dyad, or alternatively

$$\underline{J}^{\alpha/\beta} = 4\pi\lambda_{\alpha\beta}\frac{e_\alpha^2 e_\beta^2}{m_\alpha^2}\left\{f^\alpha(\underline{v})\,\nabla_{\underline{v}} H(\underline{v}) - \frac{1}{2}\nabla_{\underline{v}}\cdot\left[f^\alpha(\underline{v})\nabla_{\underline{v}}\nabla_{\underline{v}} G(\underline{v})\right]\right\}$$

where the Rosenbluth potentials are

$$G(\underline{v}) = \int f^{\beta}(\underline{v}')u\,d^3v'$$

$$H(\underline{v}) = \left(1 + \frac{m_\alpha}{m_\beta}\right)\int f^{\beta}(\underline{v}')u^{-1}d^3v' \quad .$$

If species α is a weak beam (number and energy density small compared with background) streaming through a Maxwellian plasma, then

$$\underline{J}^{\alpha/\beta} = -\nu_s^{\alpha/\beta}\,\underline{v}f^{\alpha} - \frac{1}{2}\nu_{\perp}^{\alpha/\beta}v^2\nabla_{\underline{v}}f^{\alpha}$$

$$+ \frac{1}{2}(\nu_{\perp}^{\alpha/\beta} - \nu_{\parallel}^{\alpha/\beta})\underline{v}\,\underline{v}\cdot\nabla_{\underline{v}}f^{\alpha} \quad .$$

<u>Transport coefficients</u>[18] .

Transport equations for a multispecies plasma:

$$\frac{D^{\alpha}n_\alpha}{Dt} + n_\alpha\underline{\nabla}\cdot\underline{v}_\alpha = 0 \quad ;$$

$$m_\alpha n_\alpha\frac{D^{\alpha}\underline{v}_\alpha}{Dt} = -\underline{\nabla}p_\alpha - \underline{\nabla}\cdot\underline{\underline{P}}_\alpha + Z_\alpha e n_\alpha\left[\underline{E} + \frac{1}{c}\underline{v}_\alpha \times \underline{B}\right] + \underline{R}_\alpha \quad ;$$

$$\frac{3}{2}n_\alpha k\frac{D^{\alpha}T_\alpha}{Dt} + p_\alpha\nabla\cdot\underline{v}_\alpha = -\underline{\nabla}\cdot\underline{q}_\alpha - \underline{\underline{P}}_\alpha:\underline{\nabla}\,\underline{v}_\alpha + Q_\alpha,$$

where $D^{\alpha}/Dt = \partial/\partial t + \underline{v}_{\alpha} \cdot \underline{\nabla}$ and $p_{\alpha} = (3/2)\, n_{\alpha} k T_{\alpha}$. $\underline{R}_{\alpha} = \sum_{\beta} \underline{R}_{\alpha\beta}$ and $Q_{\alpha} = \sum_{\beta} Q_{\alpha\beta}$ where $\underline{R}_{\alpha\beta}$ and $Q_{\alpha\beta}$ are respectively the momentum and energy gained by the α^{th} species through collisions with the β^{th}, $\underline{\underline{P}}_{\alpha}$ is the stress tensor, and $\underline{q}_{\alpha}$ is the heat flow.

The transport coefficients in a simple two-component (electrons and singly charged ions) plasma are tabulated below. Here $\parallel$ and $\perp$ refer to the direction of the magnetic field $\underline{B} = \underline{b}B$; $\underline{u} = \underline{v}_e - \underline{v}_i$ is the relative streaming velocity; $n_e = n_i = n$; $\underline{j} = -ne\underline{u}$ is the current; $\omega_{ce} = 1.76 \times 10^7$ B and $\omega_{ci} = (m_e/m_i)\omega_{ce}$ are the electron and ion gyrofrequencies, respectively; and the basic collisional times are taken to be

$$\tau_e = \frac{3\sqrt{m_e}\,(kT_e)^{3/2}}{4\sqrt{2\pi}\, n\lambda e^4} = 3.44 \times 10^5 \; \frac{T_e^{\,3/2}}{n\lambda} \quad ,$$

and

$$\tau_i = \frac{3\sqrt{m_i}\,(kT_i)^{3/2}}{4\sqrt{\pi}\, n\lambda e^4} = 2.09 \times 10^7 \; \frac{T_i^{\,3/2}}{n\lambda}\; \mu^{1/2} \quad .$$

In the limit of large fields ($\omega_{cj}\tau_j \gg 1$) the transport processes may be summarized as follows:

momentum transfer $\quad \underline{R}_{ei} = -\underline{R}_{ie} = \underline{R} = \underline{R}_u + \underline{R}_T$;

frictional force $\quad \underline{R}_u = ne(\underline{j}_{\perp}/\sigma_{\perp} + \underline{j}_{\parallel}/\sigma_{\parallel})$;

conductivities $\quad \sigma_{\parallel} = 2.0\, \sigma_{\perp} = 2.0 \dfrac{ne^2 \tau_e}{m_e}$;

thermal force $$\underline{R}_T = -0.71 nk\underline{\nabla}_\parallel T_e - \frac{3}{2}\frac{nk}{\omega_{ce}\tau_e}\underline{b}\times\underline{\nabla}_\perp T_e \;;$$

ion heating $$Q_i = 3\frac{m_e}{m_i}\frac{nk}{\tau_e}(T_e - T_i) \;;$$

electron heating $$Q_e = -Q_i - \underline{R}\cdot\underline{u} \;;$$

ion heat flux $$\underline{q}_i = -\kappa_\parallel^i\underline{\nabla}_\parallel kT_i - \kappa_\perp^i\underline{\nabla}_\perp kT_i + \kappa_\wedge^i\underline{b}\times\underline{\nabla}_\perp kT_i \;;$$

ion thermal conductivities $$\kappa_\parallel^i = 3.9\frac{nkT_i\tau_i}{m_i} \;;$$

$$\kappa_\perp^i = 2\frac{nkT_i}{m_i\omega_{ci}^2\tau_i} \;;$$

$$\kappa_\wedge^i = -\frac{5}{2}\frac{nkT_i}{m_i\omega_{ci}} \;;$$

electron heat flux $$\underline{q}^e = \underline{q}_u^e + \underline{q}_T^e \;;$$

frictional heat flux $$\underline{q}_u^e = 0.71\, nkT_e\underline{u}_\parallel + \frac{3}{2}\frac{nkT_e}{\omega_{ce}\tau_e}\underline{b}\times\underline{u}_\perp \;;$$

thermal gradient heat flux $$\underline{q}_T^e = -\kappa_\parallel^e\underline{\nabla}_\parallel kT_e - \kappa_\perp^e\underline{\nabla}_\perp kT_e - \kappa_\wedge^e\underline{b}\times\underline{\nabla}_\perp kT_e \;;$$

electron thermal conductivities

$$\varkappa_{\parallel}^{e} = 3.2 \frac{nkT_e\tau_e}{m_e} \;;$$

$$\varkappa_{\perp}^{e} = 4.7 \frac{nkT_e}{m_e\omega_{ce}^2\tau_e} \;;$$

$$\varkappa_{\wedge}^{e} = \frac{5}{2} \frac{nkT_e}{m_e\omega_{ce}} \;;$$

stress tensor (both species)

$$P_{xx} = -\frac{\eta_o}{2}(W_{xx} + W_{yy}) - \frac{\eta_1}{2}(W_{xx} - W_{yy}) - \eta_3 W_{xy} \;;$$

$$P_{yy} = -\frac{\eta_o}{2}(W_{xx} + W_{yy}) + \frac{\eta_1}{2}(W_{xx} - W_{yy}) + \eta_3 W_{xy} \;;$$

$$P_{xy} = P_{yx} = -\eta_1 W_{xy} + \frac{\eta_3}{2}(W_{xx} - W_{yy}) \;;$$

$$P_{xz} = P_{zx} = -\eta_2 W_{xz} - \eta_4 W_{yz} \;;$$

$$P_{yz} = P_{zy} = -\eta_2 W_{yz} + \eta_4 W_{xz} \;;$$

$$P_{zz} = -\eta_o W_{zz}$$

(here the z axis is defined parallel to $\underline{B}$) ;

ion viscosity $$\eta_o^i = 0.96\, nkT_i \tau_i \quad ;$$

$$\eta_1^i = \frac{3}{10} \frac{nkT_i}{\omega_{ci}^2 \tau_i} \quad ;$$

$$\eta_2^i = \frac{6}{5} \frac{nkT_i}{\omega_{ci}^2 \tau_i} \quad ;$$

$$\eta_3^i = \frac{1}{2} \frac{nkT_i}{\omega_{ci}} \quad ;$$

$$\eta_4^i = \frac{nkT_i}{\omega_{ci}} \quad ;$$

electron viscosity $$\eta_o^e = 0.73\, nkT_e \tau_e \quad ;$$

$$\eta_1^e = 0.51 \frac{nkT_e}{\omega_{ce}^2 \tau_e} \quad ;$$

$$\eta_2^e = 2.0 \frac{nkT_e}{\omega_{ce}^2 \tau_e} \quad ;$$

$$\eta_3^e = - \frac{1}{2} \frac{nkT_e}{\omega_{ce}} \quad ;$$

$$\eta_4^e = - \frac{nkT_e}{\omega_{ce}} \quad .$$

For both species the rate-of-strain tensor is defined as

$$W_{jk} = \frac{\partial v_j}{\partial x_k} + \frac{\partial v_k}{\partial x_j} - \frac{2}{3}\,\delta_{jk}\nabla\cdot\underline{v} \quad .$$

When $\underline{B} = 0$ the following simplifications occur:

$$\underline{R}_u = ne\underline{j}/\sigma_{\parallel} \quad ;$$

$$\underline{R}_T = -\,0.71\, n\underline{\nabla} kT_e ;$$

$$\underline{q}_i = -\,\kappa^i_{\parallel}\underline{\nabla} kT_i \quad ;$$

$$\underline{q}^e_u = 0.71\, nkT_e\underline{u} \quad ;$$

$$\underline{q}^e_T = -\,\kappa^e_{\parallel}\underline{\nabla} kT_e ;$$

$$P_{jk} = -\,\eta_o W_{jk} \quad ,$$

when $\omega_{ce}\tau_e \gg 1 \gg \omega_{ci}\tau_i$, the high-field expressions are obeyed by the electrons and the zero-field expressions by the ions.

Collisional transport theory is applicable when (1) macroscopic time rates of change satisfy $\frac{d}{dt} \ll \frac{1}{\tau}$, where τ is the longest collisional time scale, and (in the absence of a magnetic field) (2) macroscopic length scales L satisfy $L \gg \ell$, where $\ell = \bar{v}\tau$ is the mean

free path. In a strong field, $\omega_{ce}\tau \gg 1$, condition (2) is replaced by $L_{\parallel} \gg \ell$ and $L_{\perp} \gg \sqrt{\ell r_e}$ ($L_{\perp} \gg r_e$ in a uniform field), where $L_{\parallel}$ is a macroscopic scale parallel to the field $\underline{B}$ and $L_{\perp}$ is the smaller of $(\nabla_{\perp} B/B)^{-1}$ and the transverse plasma dimension. In addition, the standard transport coefficients are valid only when (3) the Coulomb logarithm satisfies $\lambda \gg 1$; (4) the electron gyroradius satisfies $r_e \gg \lambda_D$, or $B^2 \ll 8\pi n_e m_e c^2$; (5) relative drifts $\underline{u} = \underline{v}_{\alpha} - \underline{v}_{\beta}$ between two species are small compared with the thermal velocities, $u \ll (kT_{\alpha}/m_{\alpha})^{1/2}, (kT_{\beta}/m_{\beta})^{1/2}$, and (6) anomalous transport processes owing to microinstabilities are negligible.

Weakly Ionized Plasmas

Collision frequency for scattering of charged particles by neutrals

$$^{*}\nu_{\alpha} = n_o \sigma_{\alpha o}(kT_{\alpha}/m_{\alpha})^{1/2} \qquad \mathrm{sec}^{-1} \quad ,$$

where n_o is the neutral density and $\sigma_{\alpha o}$ is the cross-section, typically $\sim 5 \times 10^{-15}$ cm^2 and weakly dependent on temperature.

When the system size $L \ll \lambda_D$, the charged particle diffusion coefficients are

$$D_{\alpha} = kT_{\alpha}/m_{\alpha}\nu_{\alpha} \qquad \mathrm{cm}^2/\mathrm{sec} \ .$$

In the opposite limit, both species diffuse at the ambipolar rate

$$D_A = \frac{\mu_i D_e - \mu_e D_i}{\mu_i - \mu_e} = \frac{(T_i + T_e) D_i D_e}{T_i D_e + T_e D_i} \qquad \mathrm{cm}^2/\mathrm{sec} \quad ,$$

where $\mu_\alpha = e_\alpha/m_\alpha\nu_\alpha$ is the mobility. The conductivity σ_α satisfies $\sigma_\alpha = n_\alpha e_\alpha \mu_\alpha$.

In the presence of a magnetic field $\underline{B}$, μ and σ become tensors

$$\underline{J}^\alpha = \underline{\underline{\sigma}}^\alpha \cdot E = \sigma_\parallel^\alpha \underline{E}_\parallel + \sigma_\perp^\alpha \underline{E}_\perp + \sigma_\wedge^\alpha \underline{E} \times \underline{b} \quad ,$$

where $\underline{b} = \underline{B}/B$ and

$$\sigma_\parallel^\alpha = n_\alpha e_\alpha^2/m_\alpha \nu_\alpha \quad ;$$

$$\sigma_\perp^\alpha = \sigma_\parallel^\alpha \nu_\alpha^2/(\nu_\alpha^2 + \omega_{c\alpha}^2) \quad ;$$

$$\sigma_\wedge^\alpha = \sigma_\parallel^\alpha \nu_\alpha \omega_{c\alpha}/(\nu_\alpha^2 + \omega_{c\alpha}^2) \quad .$$

Here $\sigma_\wedge$ and $\sigma_\perp$ are the Hall and Pedersen conductivities, respectively.

IONOSPHERIC PARAMETERS [19]

These are typical nighttime values. Where two numbers are entered, the first refers to the lower and the second to the upper portion of the layer.

Quantity	E Region	F Region
Number density (m^{-3})	1.5×10^{10} - 3.0×10^{10}	5×10^{10} - 2×10^{11}
Height-integrated number density (m^{-2})	9×10^{14}	4.5×10^{15}
Ion-neutral collision frequency (sec^{-1})	2×10^{3} - 10^{2}	0.5 - .05
Ion gyro-/collision frequency ratio $\varkappa_i$	.09 - 2.0	4.6×10^{2} - 5.0×10^{3}
Ion Pedersen conductance $\varkappa_i/(1 + \varkappa_i^2)$	0.09 - 0.5	2.2×10^{-3} - 2×10^{-4}
Ion Hall conductance $\varkappa_i^2/(1 + \varkappa_i^2)$	8×10^{-4} - 0.8	1.0
Electron-neutral collision frequency	1.5×10^{4} - 9.0×10^{2}	80 - 10
Electron gyro-/collision frequency ratio $\varkappa_e$	4.1×10^{2} - 6.9×10^{3}	7.8×10^{4} - 6.2×10^{5}
Electron Pedersen conductance $\varkappa_e/(1 + \varkappa_e^2)$	2.7×10^{-3} - 1.5×10^{-4}	10^{-5} - 1.5×10^{-6}
Electron Hall conductance $\varkappa_e^2/(1 + \varkappa_e^2)$	1.0	1.0
Mean molecular weight	28 - 26	22 - 16
Ion gyrofrequency (sec^{-1})	180 - 190	230 - 300*
Neutral diffusion coefficient (m^2/sec)	30 - 5×10^{3}	10^{5}

The terrestrial magnetic field in the lower ionosphere at equatorial latitudes is approximately $B_o = .35 \times 10^{-4}$ tesla. The earth's radius is $R_E = 4.0 \times 10^4/2\pi$ km $\approx 6.4 \times 10^3$ km.

CTR[20]

Natural abundance of deuterium $n_D/n_H = 1.5 \times 10^{-4}$

Mass ratios

$$m_e/m_D = 2.72 \times 10^{-4} = 1/3670$$

$$(m_e/m_D)^{\frac{1}{2}} = 1.65 \times 10^{-2} = 1/60.6$$

$$m_e/m_T = 1.82 \times 10^{-4} = 1/5496$$

$$(m_e/m_T)^{\frac{1}{2}} = 1.35 \times 10^{-2} = 1/74.1$$

Fusion reactions (branching ratios are correct for energies near the cross-section peaks; a negative yield means the reaction is endothermic).[21]

(1a)	$D + D \xrightarrow[50\%]{}$	$T(1.01\ MeV) + p(3.02\ MeV)$
(1b)	$\xrightarrow[50\%]{}$	$He^3(.82\ MeV) + n(2.45\ MeV)$
(2)	$D + T \longrightarrow$	$He^4(3.5\ MeV) + n(14.1\ MeV)$
(3)	$D + He^3 \longrightarrow$	$He^4(3.6\ MeV) + p(14.7\ MeV)$
(4)	$T + T \longrightarrow$	$He^4 + 2\ n + 11.3\ MeV$
(5a)	$He^3 + T \xrightarrow[51\%]{}$	$He^4 + p + n + 12.1\ MeV$
(5b)	$\xrightarrow[43\%]{}$	$He^4(4.8\ MeV) + D(9.5\ MeV)$
*(5c)	$\xrightarrow[6\%]{}$	$He^5(2.4\ MeV) + p(11.9\ MeV)$
(6)	$p + Li^6 \longrightarrow$	$He^4(1.7\ MeV) + He^3(2.3\ MeV)$
(7a)	$p + Li^7 \xrightarrow[\sim 20\%]{}$	$2\ He^4 + 17.3\ MeV$
(7b)	$\xrightarrow[\sim 80\%]{}$	$Be^7 + n - 1.6\ MeV$
(8)	$D + Li^6 \longrightarrow$	$2\ He^4 + 22.4\ MeV$
(9)	$p + B^{11} \longrightarrow$	$3\ He^4 + 8.7\ MeV$

The rates for the principal reactions, averaged over Maxwellian distributions, are [21, 22]

Temperature (keV)	$\overline{\sigma v}$ (cm^3/sec)				
	D - D (1a) + (1b)	D - T (2)	D - He^3 (3)	T - T (4)	T - He^3 (5a) + (5b) + (5c)
1.0	1.5×10^{-22}	5.5×10^{-21}	3×10^{-26}	3.3×10^{-22}	10^{-28}
2.0	5.4×10^{-21}	2.6×10^{-19}	1.4×10^{-23}	7.1×10^{-21}	10^{-25}
5.0	1.8×10^{-19}	1.3×10^{-17}	6.7×10^{-21}	1.4×10^{-19}	2.1×10^{-22}
10.0	1.2×10^{-18}	1.1×10^{-16}	2.3×10^{-19}	7.2×10^{-19}	1.2×10^{-20}
20.0	5.2×10^{-18}	4.2×10^{-16}	3.8×10^{-18}	2.5×10^{-18}	2.6×10^{-19}
50.0	2.1×10^{-17}	8.7×10^{-16}	5.4×10^{-17}	8.7×10^{-18}	5.3×10^{-18}
100.0	4.5×10^{-17}	8.5×10^{-16}	1.6×10^{-16}	1.9×10^{-17}	2.7×10^{-17}
200.0	8.8×10^{-17}	6.3×10^{-16}	2.4×10^{-16}	4.2×10^{-17}	9.2×10^{-17}
500.0	1.8×10^{-16}	3.7×10^{-16}	2.3×10^{-16}	8.4×10^{-17}	2.9×10^{-16}
1000.0	2.2×10^{-16}	2.7×10^{-16}	1.8×10^{-16}	8.0×10^{-17}	5.2×10^{-16}

The total cross section as a function of E, the energy in keV of the incident particle [the first ion on the left side of Eqs. (1-5)], assuming the target ion at rest, can be fitted by[21]

$$* \qquad \sigma_T(E) = \frac{A_5 + \left[(A_4 - A_3 E)^2 + 1\right]^{-1} A_2}{E\left[\exp\left(A_1/\sqrt{E}\right) - 1\right]}$$

where the Duane coefficients A_j for the principal fusion reactions are as follows:

Coefficient (units)	Reaction (Eq. Number)					
	D - D (1a)	D - D (1b)	D - T (2)	D - He^3 (3)	T - T (4)	T - He^3 (5)
$A_1\ (keV)^{1/2}$	46.097	47.88	45.95	89.27	38.39	123.1
A_2 (keV-barn)	372	482	5.02×10^4	2.59×10^4	448	1.125×10^4
$A_3\ (keV)^{-1}$	4.36×10^{-4}	3.08×10^{-3}	1.368×10^{-2}	3.98×10^{-3}	1.02×10^{-3}	0
A_4	1.220	1.177	1.076	1.297	2.09	0
A_5 (keV-barn)	0	0	409	647	0	0

For low energies ($T \leq 25$ keV) the data may be represented by

$$(\overline{\sigma v})_{DD} = 2.33 \times 10^{-14} T^{-2/3} \exp(- 18.76\, T^{-1/3}) \ \ \mathrm{cm^3/sec}$$

$$(\overline{\sigma v})_{DT} = 3.68 \times 10^{-12}\, T^{-2/3} \exp(- 19.94\, T^{-1/3}) \ \ \mathrm{cm^3/sec}$$

The power density released in the form of charged particles is

$$P_{DD} = 3.3 \times 10^{-13}\, n_D^2 (\overline{\sigma v})_{DD} \ \mathrm{watt/cm^3} \text{ (including subsequent D-T reaction)}$$

$$P_{DT} = 5.6 \times 10^{-13}\, n_D n_T (\overline{\sigma v})_{DT} \ \mathrm{watt/cm^3}$$

$$P_{DHe^3} = 2.9 \times 10^{-12}\, n_D n_{He^3} (\overline{\sigma v})_{DHe^3} \ \mathrm{watt/cm^3}$$

The curie (abbreviated Ci) is a measure of radioactivity: 1 curie = 3.7×10^{10} counts/sec. Absorbed radiation dose is measured in rads: 1 rad = 10^2 erg/g.

RELATIVISTIC ELECTRON BEAMS

Here $\gamma = (1 - \beta^2)^{-\frac{1}{2}}$ is the relativistic scaling factor; in analytic formulas units are mks or cgs, as indicated; in numerical formulas, I is in amps, B in gauss, electron density N in cm^{-1}, and temperature, voltage and energy in MeV ; $\beta_z = v_z/c$.

Relativistic electron gyroradius $\quad r_e = \frac{mc^2}{eB}(\gamma^2 - 1)^{\frac{1}{2}}$ (cgs)

$$= 1.70 \times 10^3 (\gamma^2 - 1)^{\frac{1}{2}} B \quad \text{cm}$$

Relativistic electron energy $\quad W = mc^2\gamma$ (cgs) $= 0.511\ \gamma$ MeV

Alfvén-Lawson limit $\quad I_A = \frac{mc^3}{e}\beta_z\gamma$ (cgs)

$$= \frac{4\pi}{\mu_o}\frac{mc}{e}\beta_z\gamma \quad \text{(mks)}$$

$$= 1.70 \times 10^4\ \beta_z\gamma \quad \text{amp}$$

The ratio of net current to I_A is

$$I/I_A = \nu/\gamma,$$

where $\nu = Nr_e$, with $r_e = e^2/mc^2 = 2.82 \times 10^{-13}$ cm. Beam electron number density is

$$n = 2.08 \times 10^8\ J/\beta\ \text{cm}^{-3},$$

where J is current density in amp/cm^2. For a uniform beam of radius a (in cm),

$$n = 6.63 \times 10^{7}\ I/\beta a^{2}\ \mathrm{cm}^{-3},$$

ıd

$$2\ r_e/a = \nu/\gamma.$$

ıturated parapotential current (magnetically self-limited flow along ɥuipotentials in pinched diodes and transmission lines)[23]

$$I_p = 8.5 \times 10^{3}\ G\gamma\ \ell n \left[\gamma + (\gamma^{2} - 1)^{1/2}\right] \quad \text{amp},$$

ıere G is a geometrical factor depending on the diode structure:

$\dfrac{\gamma}{2\pi d}$ for parallel plane cathode and anode of width γ, separation d;

$\left[\ell n \dfrac{R_2}{R_1}\right]^{-1}$ for cylinders of radii R_1 (inner) and R_2 (outer);

$\dfrac{R_c}{d_o}$ for conical cathode of radius R_c, maximum separation d_o (at $r = R_c$) from plane anode, etc.

e $\beta \to 0$ ($\gamma \to 1$), both I_A and I_p vanish.

...pinch condition

$$I^{2} = 2Nk(T_e + T_i)c^{2} \quad \text{(cgs)}$$

$$= 3.20 \times 10^{-4}\ N(T_e + T_i) \quad \text{amp}^{2}$$

Child's law (space-charge limited current density between parallel plates with voltage drop V in MeV and separation d in cm)

Nonrelativistic: $$J = 2.34 \times 10^3 \, V^{3/2} \, d^{-2} \, \frac{\text{amp}}{\text{cm}^2}$$

Relativistic ($\gamma \geq 2$): $$J = 2.71 \times 10^3 \, (\gamma^{1/2} - 0.8471)^2 \, d^{-2} \, \frac{\text{amp}}{\text{cm}^2} \, .$$

Voltage registered by Rogowski coil of minor cross-sectional area A, n turns, major radius a, inductance L, external resistance R and capacitance C (all in mks):

self-integrating $$V = \frac{R}{L} \frac{nA\mu_o I}{2\pi a} \quad \text{volts}$$

externally integrated $$V = \frac{1}{RC} \frac{nA\mu_o I}{2\pi a} \quad \text{volts}$$

X-ray production for target with average atomic number Z($V \leq 5$ MeV)

$$\eta = \frac{\text{X-ray power}}{\text{beam power}} = 7 \times 10^{-4} \, ZV$$

X-ray dose at 1 meter generated by an e-beam depositing total charge Q coulombs while $V \geq .84 \, V_{max}$ in material with charge state Z

$$D = 150 \, V_{max}^{2.8} \, Q \, Z^{1/2} \quad \text{rads}$$

ELECTROSTATIC STREAMING INSTABILITIES [24]

In this table, subscripts e,i,d,b stand for "electron," "ion," "drift," and "beam," respectively. Thermal velocities are denoted by a bar. In addition, the following are used:

m	electron mass
M	ion mass
V	velocity
T	temperature
n_e, n_i	number density
n	harmonic number
$C_s = T_e/M$	ion sound speed
ω_e, ω_i	plasma frequency
λ_D	Debye length
r_e, r_i	gyroradius
β	plasma/magnetic energy density ratio
V_A	Alfvén speed
Ω_e, Ω_i	gyro frequency
Ω_H	hybrid gyro frequency, $\Omega_H^2 = \Omega_e \Omega_i$
U	relative drift velocity of two ion species

Name	Conditions	Parameters of Most Unstable Mode: Growth Rate	Frequency	Wave Number	Group Velocity	Saturation Mechanism
Electron - electron	$V_d > V_{ej},\ j = 1,2$	$\frac{1}{2}\,\omega_e$	0	$\frac{\sqrt{3}}{2}\,\omega_e/V_d$	0	Electron trapping until $\bar{V}_e \sim V_d$
Beam-plasma	$(n_b/n_p)^{\frac{1}{3}} > \bar{V}_b/V_b$	$\frac{\sqrt{3}}{2^{4/3}}\left(\frac{n_b}{n_p}\right)^{1/3}\omega_e$	$\omega_e\left[1 - \frac{1}{2^{4/3}}\,(n_b/n_p)^{\frac{1}{3}}\right]$	ω/V_d	$\frac{2}{3}\,V_b$	Trapping of beam electrons
Buneman	$V_d > (M/m)^{\frac{1}{3}}\,\bar{V}_i$, $V_d > \bar{V}_e$	ω_e/V_d	$\frac{1}{2^{4/3}}\,\omega_e(m/M)^{\frac{1}{3}}$	ω_e/V_d	$\frac{2}{3}\,V_d$	Electron trapping until $\bar{V}_e \sim V_d$
Weak beam-plasma	$\bar{V}_b/V_b > (n_b/n_p)^{\frac{1}{3}}$	$\frac{1}{2}\,\frac{n_b}{n_p}\left(\frac{V_b}{\bar{V}_b}\right)^2\omega_e$	ω_e	ω_e/V_b	$3\bar{V}_e^2/V_b$	Quasilinear or non-linear
Beam-plasma (hot electron)	$\bar{V}_e > V_b > \bar{V}_b$, $(n_b/n_p)^{\frac{1}{2}}\,\bar{V}_e > \bar{V}_b$	$\left(\frac{n_b}{n_p}\right)^{\frac{1}{2}}\frac{\bar{V}_e}{V_b}\,\omega_e$	$\frac{V_b}{V_e}\,\omega_e$	λ_D^{-1}	V_b	Quasilinear or non-linear
Ion acoustic	$T_e \gg T_i$, $V_d > C_s$	$\left(\frac{m}{M}\right)^{\frac{1}{2}}\omega_i$	$\frac{C_s}{V_e}\,\omega_e$	λ_D^{-1}	C_s	Quasilinear; ion tail formation; non-linear scattering; or resonance broadening.

Name	Condition	Parameters of Most Unstable Mode				
		Growth Rate	Frequency	Wave Number	Group Velocity	
Anisotropic temperature (Hydro)	$\frac{T^e_{\parallel}}{T^e_{\perp}} < \frac{1}{2}$	Ω_e	$\omega_e \cos\theta \sim \Omega_e$	r_e^{-1}	$\bar{V}_{e\perp}$	Isotropization
Ion Cyclotron	$V_d > 20\,\bar{V}_i$	$0.1\,\Omega_i$	$1.2\,\Omega_i$	r_i^{-1}	$\frac{1}{3}\,\bar{V}_i$	Ion heating
Beam Cyclotron (Hydro)	$V_d > C_s$	$\frac{1}{\sqrt{2}}\Omega_e$	$n\,\Omega_e$	$\frac{1}{\sqrt{2}}\lambda_D^{-1}$	$V_d \lesssim V_g \lesssim C_s$	Resonance broadening
Modified two stream (Hydro)	$\sqrt{1+\beta}\,V_A > V_d > C_s$	$\frac{1}{2}\,\Omega_H$	$\frac{\sqrt{3}}{2}\,\Omega_H$	$\sqrt{3}\,\frac{\Omega_H}{V_d}$	$\frac{1}{2}\,V_d$	Trapping
Ion-Ion	$U < 2V_A\sqrt{1+\beta}$	$\frac{1}{2\sqrt{2}}\,\Omega_H$	0	$\sqrt{3/2}\,\frac{\Omega_H}{U}$	0	Ion trapping
Ion-Ion	$U < 2\,C_s$	$\frac{1}{2\sqrt{2}}\,\omega_i$	0	$\sqrt{3/2}\,\frac{\omega_i}{U}$	0	Ion trapping

LASERS

Parameters of Important Laser Systems

Efficiencies and power levels are approximately state-of-the-art (1976)

Type	Wavelength (microns)	Efficiency	Power levels available (W)	
			Pulsed	CW
CO_2	10.6	.01 - .02 pulsed	$> 10^{11}$	$> 10^{5}$
CO	5	0.4	$> 10^{9}$	> 100
Iodine	1.315	3×10^{-3}	$> 10^{11}$	—
Nd-glass, YAG	1.06	10^{-3}	$> 10^{12}$	1 - 300
Ruby	.6943	$< 10^{-3}$	10^{10}	1
He - Ne	.6328	10^{-4}	—	$1 - 50 \times 10^{-3}$
Argon ion	.45 - .60	10^{-3}	5×10^{4}	1 - 10
N_2	.3371	10^{-3} - .05	$10^{5} - 10^{6}$	—
Xenon	.175	.02	$> 10^{8}$	—

3. Ponderomotive force

$$\underset{\sim}{f} = N \underset{\sim}{\nabla} \langle E^2 \rangle / 8\pi N_c$$

where

$$N_c = 1.1 \times 10^{21}/\lambda_o^2 \text{ cm}^{-3} .$$

4. For uniform illumination of a lens with f-number F, the diameter d at focus (85% of the energy) and the depth of focus ℓ (first zero in intensity) are given by

$$d \approx 2.44 \, F\lambda_o \theta/\theta_{DL} \; \mu \text{ and } \ell \approx \pm 2 \, F^2 \lambda_o \theta/\theta_{DL} \; \mu .$$

Here θ is the beam divergence containing 85% of energy and θ_{DL} is the diffraction limited divergence:

$$\theta_{DL} = \frac{2.44 \, \lambda_o}{b} \qquad \text{(aperture b in } \mu)$$

These formulas are modified for nonuniform illumination of the lens (such as gaussian) or for pathological laser profiles.

Formulas

1. An e-m wave with $\underline{k} \parallel \underline{B}$ has an index of refraction given by

$$n_{\pm} = \left[1 - \frac{\omega_{pe}^2}{\omega(\omega \mp \omega_{ce})} \right]^{1/2}$$

where $\pm$ refers to the velocity. The rate of change of polarization angle θ as a function of displacement s (Faraday rotation) is given by

$$\frac{d\theta}{ds} = \frac{1}{2} k(n_- - n_+) = 2.36 \text{ NBf}^{-2} \text{ cm}^{-1} ,$$

where N is the electron number density, B is the field strength, and f is the wave frequency. For $N = 10^{20}$ cm^{-3}, B = 10 kG and $f = 3 \times 10^{14}$ Hz (corresponding to a wavelength $\lambda = 1\ \mu$), $\frac{d\theta}{ds} = 2.6 \times 10^{-3}$ rad/m.

2. The quiver velocity of an electron in an e-m field of angular frequency ω is

$$v_o = \frac{eE_{max}}{m\omega} = 25.6 \sqrt{I}\, \lambda_o \text{ cm/sec}$$

in terms of the laser flux $I = \frac{cE^2_{max}}{8\pi}$, with I in watts/cm^2, laser wavelength λ_o in microns. The ratio of quiver energy to thermal energy is

$$\frac{m_e v_o^2}{2\ kT} = 1.81 \times 10^{-13} \frac{\lambda_o^2 I}{T} ,$$

T in eV. E.g., if $I = 10^{15}$ watts/cm^2, $\lambda_o = 1\ \mu$, T = 2 keV, then

$W_{quiver}/W_{thermal} \approx .1$

ATOMIC AND RADIATION PROCESSES

The subject of atomic physics is central and indispensible for unstanding the behavior of real plasmas. It is ill-suited for distillation into a few general formulas, however. Those compiled here should not be applied blindly. The inexperienced user would do well to first consult the references cited, particularly [31] and [35], which are more accessible to plasma physicists.

Energies and temperatures are in eV; all other units are cgs except where noted. Z is the charge state (Z = 0 refers to a neutral atom); the subscript e labels electrons. N refers to number density, n to principal quantum number. Asterisk superscripts on level population densities denote local thermodynamic equilibrium (LTE) values. Thus N_n^* is the LTE number density of atoms (or ions) in level n.

Characteristic atomic collision cross section

(1) $$\pi a_0^2 = 8.80 \times 10^{-17}\ \text{cm}^2$$

Binding energy for outer electron in level labelled by quantum numbers n, ℓ

(2) $$E_\infty^Z(n,\ell) = -\frac{Z^2 E_\infty^H}{(n-\Delta_\ell)^2}\ \text{eV},$$

where $E_\infty^H = 13.6$ eV is the hydrogen ionization energy and $\Delta_\ell = .75\ell^{-5}$, $\ell \geq 5$, is the quantum defect.

Excitation and Decay

Cross section (Bethe approximation) for electron excitation by dipole allowed transition $m \rightarrow n$ [25,26]

$$\sigma_{mn} = 2.36 \times 10^{-13} \frac{f_{nm} g(n,m)}{\epsilon \Delta E_{nm}} \text{ cm}^2, \tag{3}$$

where f_{nm} is the oscillator strength, $g(n,m)$ is the Gaunt factor, ϵ is the incident electron energy, and $\Delta E_{nm} = E_n - E_m$.

Electron excitation rate averaged over Maxwellian velocity distribution [27,28]

$$X_{mn} = N_e < \sigma_{mn} v > = \frac{1.6 \times 10^{-5} f_{nm} \langle g(n,m) \rangle N_e}{\Delta E_{nm} \sqrt{T_e}} \exp\left(- \frac{\Delta E_{nm}}{T_e} \right) \text{sec}^{-1}, \tag{4}$$

where $< g(n,m)>$ denotes the thermal averaged Gaunt factor whose behavior as a function of energy is shown in Fig. 3 (generally ~ 1 for atoms, ~ 0.2 for ions).

Rate for electron collisional de-excitation

$$Y_{nm} = (N^*_m / N^*_n) X_{mn} \text{sec}^{-1}. \tag{5}$$

Here $N^*_m / N^*_n = (g_m / g_n) \exp(\Delta E_{nm} / T_e)$ is the Boltzmann relation for level population densities, where g_n is the statistical weight of level n.

Rate for spontaneous radiative decay $n \rightarrow m$ (Einstein A coefficient) [27]

$$A_{nm} = 4.3 \times 10^7 \frac{g_n}{g_m} f_{nm} (\Delta E_{nm})^2 \text{sec}^{-1} . \tag{6}$$

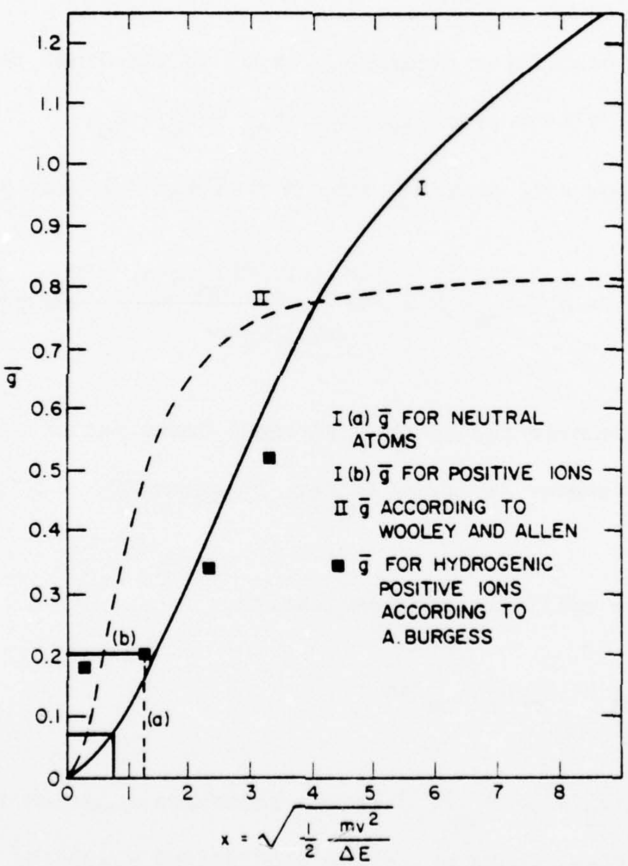

Fig. 3 — The effective Gaunt factor $\bar{g}$ for neutral atoms and positive ions. For neutrals the curve is deduced from the experimental results for $1s \rightarrow 2p$ in H.

Intensity emitted per unit volume from the transition $n \rightarrow m$ in an optically thin plasma

(7) $$I_{nm} = 1.6 \times 10^{-19} A_{nm} N_n \Delta E_{nm} \ W/cm^3.$$

Condition for steady state in a corona model

(8) $$N_o N_e < \sigma_{on} v > = N_n A_{no} ,$$

where the ground state is labelled by a subscript zero.

Hence for a transition $n \rightarrow m$ in ions, where $< g(n,m) > \approx 0.2$,

(9) $$I_{nm} = 5.1 \times 10^{-25} f_{nm} (g_o/g_m)(\Delta E_{nm}/\Delta E_{no})^3 N_e N_o T_e^{-1/2} \exp(-\Delta E_{no}/T_e) \ W/cm^3.$$

Ionization and Recombination

In a general time-dependent situation the number density of the charge state Z satisfies

(10) $$\frac{dN(Z)}{dt} = N_e [- S(Z)N(Z) - \alpha(Z)N(Z) + S(Z - 1)N(Z - 1) + \alpha(Z + 1)N(Z + 1)].$$

Here $S(Z)$ is the ionization rate. The recombination rate $\alpha(Z)$ has the form $\alpha(Z) = \alpha_r(Z) + N_e \alpha_3(Z)$, where α_r and α_3 are the radiative and three-body recombination rates, respectively.

Classical ionization cross-section for any atomic shell k [29]

(11) $$\sigma_i = 6 \times 10^{-14} \frac{b_k}{U_k^2} g_k(x) \ cm^2;$$

Here b_k is the number of shell electrons, U_k is the binding energy of the ejected electron, $x = \epsilon/U_k$ where ϵ is the incident electron energy, and g is a universal function with a maximum value ≈ 0.2 at $x \approx 4$.

Ionization from ion ground state, averaged over Maxwellian electron distribution[28]

$$(12) \qquad S(Z) = \frac{10^{-5}(T_e/E_\infty^Z)^{1/2}}{(E_\infty^Z)^{3/2}(6 + T_e/E_\infty^Z)} \exp\left(-E_\infty^Z/T_e\right) \text{ cm}^3/\text{sec},$$

where E_∞^Z is the ionization energy.

Electron-ion radiative recombination rate $(e + N(Z) \rightarrow N(Z-1) + h\nu)$[30]

$$(13) \qquad \alpha_r(Z) = 5.2 \times 10^{-14}\, Z \sqrt{E_\infty^Z/T_e} \left\{ 0.43 + \frac{1}{2} \ln\left(E_\infty^Z/T_e\right) + 0.469 \left(E_\infty^Z/T_e\right)^{-1/3} \right\} \text{ cm}^3/\text{sec}.$$

For 10 keV $< T_e/Z^2 <$ 150 keV, this becomes approximately[28]

$$(14) \qquad \alpha_r(Z) = 2.7 \times 10^{-13} \frac{Z^2}{\sqrt{T_e}} \text{ cm}^3/\text{sec}.$$

Collisional (three-body) recombination rate for singly ionized plasma[31]

$$*(15) \qquad \alpha_3 = 8.75 \times 10^{-27}\, T_e^{-9/2} \text{ cm}^6/\text{sec}$$

(limited to low temperatures, $kT_e < E_\infty^H$). Photoionization cross

section for ions in level n, ℓ (short wavelength limit)[24]

(16) $$\sigma_{ph}(n, \ell) = 1.64 \times 10^{-16} Z^5/(n^3 K^{7+2\ell}),$$

where K is wavenumber in Rydberg units (1 Rydberg = 1.0974×10^5 cm^{-1}).

Charge Exchange

Two kinds of charge exchange take place in a plasma, depending on ΔE in

(17) $$A + B^+ \rightarrow A^+ + B + \Delta E .$$

When $\Delta E = 0$ (the internal energy difference vanishes) the process is called resonant. When A = B the resonance is symmetric; otherwise it is asymmetric. When $\Delta E \neq 0$ the process (17) is just a charge transfer reaction. In this case the adiabatic principle of Massey suggests that the peak of the cross section σ arises when

$$V \simeq \ell \frac{\Delta E}{h} .$$

Here ℓ is the interaction range and V is a critical relative velocity. For larger values of V,

(18) $$\sigma \approx k \exp\left(- \frac{\ell|\Delta E|}{4 hV}\right),$$

with k being a constant for each reaction. However, for symmetric charge exchange the cross-section as given by Fursov can be expressed as[32]

(19) $$\sigma = \left[a - b \log_{10} E\right]^2 \quad cm^2 .$$

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Values of a and b are given below for some selected reactions[32] for the energy range $E_{min} < E < E_{max}$

Process	a (10^{-8} cm)	b (10^{-8} cm)	E_{min} (eV)	E_{max} (eV)
$H^+ — H$	7.6	1.06	20	10,000
$He^+ — He$	4.7	1.1	20	6,000
$Ne^+ — Ne$	6.1	1.7	20	6,000
$H^+ — O$	4.2	0.38	25	8,000

Equilibrium Models

Saha equilibrium [33]

$$(20) \qquad \frac{N_e N_1^*(Z)}{N_n^*(Z-1)} = 6.0 \times 10^{21} \frac{g_1^Z T_e^{3/2}}{g_n^{Z-1}} \exp\left(-\frac{E_\infty^Z(n,\ell)}{T_e}\right) \mathrm{cm}^{-3},$$

where g_n^Z is the statistical weight for level n of charge state Z and $E_\infty^Z(n,\ell)$ is the ionization energy of the neutral atom initially in level (n,ℓ), given by Eq. (2).

In a steady state at high electron density

$$(21) \qquad \frac{N_e N^*(Z)}{N^*(Z-1)} = \frac{S(Z-1)}{\alpha_3} \mathrm{cm}^{-3},$$

a function only of T.

Conditions for LTE

(a) Collisional and radiative excitation rates for a level n must satisfy [33]

(22) $$Y_{nm}/A_{nm} > 10.$$

(b) Electron density must satisfy[33]

(23) $$N_e \geq 7 \times 10^{18} Z^7 n^{-17/2} (T/E_{\infty}^{Z})^{1/2} \text{ cm}^{-3}.$$

Steady state condition in corona model

(24) $$\frac{N(Z-1)}{N(Z)} = \frac{\alpha_r}{S(Z-1)}$$

Corona model is applicable if[34]

(25) $$10^{12}\, t_I^{-1} < N_e < 10^{16} T_e^{7/2} \text{ cm}^{-3},$$

where t_I^{-1} is the inverse ionization time.

Radiation[†]

Average radiative decay rate of state with principal quantum number n is

(26) $$A_n = \sum_{m<n} A_{nm} = 1.6 \times 10^{10}\, Z^4 n^{-9/2}\ \text{sec}^{-1}$$

Natural line width (ΔE in eV)

(27) $$\Delta E\ \Delta t = h = 4.14 \times 10^{-15}\ \text{eV-sec.}$$

where Δt is the lifetime of the line.

Doppler width

(28) $$\frac{\Delta\lambda}{\lambda} = 7.7 \times 10^{-5} \sqrt{\frac{T}{\mu}}\ ,$$

where μ is the emitting atom or ion mass in units of the proton mass.

Optical depth for a Doppler broadened line [33]

(29) $$\tau = 1.76 \times 10^{-13}\, \lambda \left(\frac{Mc^2}{kT}\right)^{1/2} N\ell = 5.4 \times 10^{-9}\, \lambda \left(\frac{\mu}{T}\right)^{1/2} N\ell,$$

where λ is wavelength and ℓ the physical depth of the plasma; M, N and T are atomic mass, number density and temperature of the absorber; μ is M divided by the proton mass. Optically thin means $\tau < 1$.

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† Note that the units employed here are those summarized on p. 88.

Resonance absorption cross section at center of line

$$\sigma_{\lambda=\lambda_c} = 5.6 \times 10^{-13} \frac{\lambda^2}{\Delta\lambda} \text{ cm}^2 . \tag{30}$$

Wien displacement law: wavelength of maximum black body emission is given by

$$\lambda_{max} = 2.50 \times 10^{-5} T^{-1} \text{ cm}. \tag{31}$$

Radiation from surface of black body of temperature T

$$W = 1.03 \times 10^5 T^4 \text{ watts/cm}^2. \tag{32}$$

Bremsstrahlung from hydrogen-like plasma [19]

$$P_{Br} = 1.69 \times 10^{-32} N_e T_e^{1/2} \Sigma \left[Z^2 N(Z)\right] \text{ watts/cm}^3, \tag{33}$$

where the sum is over all ionization states Z.

Bremsstrahlung optical depth [35]

$$\tau = 5.0 \times 10^{-38} N_e N_i Z^2 \bar{g} \ell T^{-7/2}, \tag{34}$$

where $\bar{g} \approx 1.2$ is an average Gaunt factor and ℓ is the physical path length.

Recombination (free-bound) radiation

$$(35) \qquad P_r = 1.69 \times 10^{-32} N_e T_e^{1/2} \Sigma \left[Z^2 N(Z) \left(\frac{E_\infty^{Z-1}}{T_e} \right) \right] \text{watts/cm}^3.$$

Cyclotron radiation in magnetic field B[19]

$$(36) \qquad P_c = 6.21 \times 10^{-21} B^2 N_e T_e \text{ watt/cm}^3$$

For $N_e kT_e = N_i kT_i = B^2/16\pi$ ($\beta = 1$, isothermal plasma),[19]

$$(37) \qquad P_c = 5.00 \times 10^{-31} N_e^2 T_e^2 \text{ watt/cm}^3.$$

Cyclotron radiation e-folding time for a single electron[35]

$$(38) \qquad t \approx \frac{9.0 \times 10^8 B^{-2}}{2.5 + \gamma} \text{ sec}^{-1},$$

where γ is the energy divided by mc^2.

Number of cyclotron harmonics trapped in a medium of finite depth ℓ[35]

$$(39) \qquad m^* = (57 \beta B\ell)^{1/6},$$

where $\beta = NkT/8\pi B^2$.

Line radiation is given by summing Eq. 9 over all species in the plasma.

REFERENCES

Most of the material in this report is quite well-known and for all practical purposes is in the "public domain." The books and articles cited below are intended primarily not for the purpose of giving credit to the original workers, but (i) to guide the reader to sources containing additional related material and (ii) to indicate where derivations, explanations, examples, etc., omitted from this compilation can be found.

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